Principles of Analysis
Lecture Notes Fall 2003

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Preface

This document consists of lecture notes made for a course in Principles of Analysis (SAU MATH 1023), taught in the fall of 2003. The textbook in use at the time was [**Ga98**].

CHAPTER 1

Lecture 1 - Sets

1. Sets

Set and *element* are undefined terms, except to the extent that we know the relationship between them is *containment*; elements are contained in sets.

If two symbols a and b represent the same element, we write $a = b$. If the symbols a and b represent different elements, we write $a \neq b$. If an element a is a contained in a set A, this relation is written $a \in A$. If a is not in A, this fact is denoted $a \notin A$. We assume that the statements $a \in A$ and $a = b$ are always either true or false, although we may not know which.

Two sets are considered equal when they contain the same elements:

$$A = B \Leftrightarrow [x \in A \Leftrightarrow x \in B].$$

2. Subsets

Let A and B be sets. We say that B is a *subset* of A and write $A \subset B$ if $x \in B \Rightarrow x \in A$.

It is clear that $A = B$ if and only if $A \subset B$ and $B \subset A$.

A set with no elements is called an *empty set*. Since two sets are equal if and only if they contain the same elements, there is only one empty set, and it is denoted $\varnothing$. The empty set is a subset of any other set.

If X is any set and $p(x)$ is a proposition whose truth or falsehood depends on each element $x \in X$, we may construct a new set consisting of all of the elements of X for which the proposition is true; this set is denoted:

$$\{x \in X \mid p(x)\}.$$

3. Set Operations

Let X be a set and let $A, B \subset X$.

The *intersection* of A and B is denoted by $A \cap B$ and is defined to be the set containing all of the elements of X that are in both A and B:

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}.$$

The *union* of A and B is denoted by $A \cup B$ and is defined to be the set containing all of the elements of X that are in either A or B:

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}.$$

We note here that there is no concept of "multiplicity" of an element in a set; that is, if x is in both A and B, then x occurs only once in $A \cup B$.

The *complement* of A with respect to B is denoted $A \setminus B$ and is defined to be the set containing all of the elements of A which are not in B:

$$A \setminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

The *symmetric difference* of A and B is denoted $A \triangle B$ and is defined to be the set containing all of the elements X which are in either A or B both not both:

$$A \triangle B = \{x \in X \mid x \in A \cup B \text{ and } x \notin A \cap B\}.$$

Proposition 1.1. *Let X be a set and let $A, B, C \subset X$. Then*

- $A = A \cup A = A \cap A$;
- $\varnothing \cap A = \varnothing$;
- $\varnothing \cup A = A$;
- $A \subset B \Leftrightarrow A \cap B = A$;
- $A \subset B \Leftrightarrow A \cup B = B$;
- $A \cap B = B \cap A$;
- $A \cup B = B \cup A$;
- $(A \cap B) \cap C = A \cap (B \cap C)$;
- $(A \cup B) \cup C = A \cup (B \cup C)$;
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$;
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;
- $A \subset B \Rightarrow A \cup (B \setminus A) = B$;
- $A \subset B \Rightarrow A \cap (B \setminus A) = \varnothing$;
- $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$;
- $(A \setminus B) \setminus C = A \setminus (B \cup C)$;
- $A \triangle B = (A \cup B) \setminus (A \cap B)$;
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

4. Cartesian Product of Two Sets

If a and b are elements, we can construct a new element

$$(a, b) = \{\{a\}, \{a, b\}\},$$

called an *ordered pair*. Ordered pairs obey the "defining property":

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

If (a, b) is an ordered pair, then a is called the *first coordinate* and b is called the *second coordinate*.

Let A and B be sets. The *cartesian product* of A and B is denoted $A \times B$ and is defined to be the set of ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Proposition 1.2. *Let X be a set and let $A, B, C \subset X$. Then*

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

5. Functions

Let A and B be sets. A *function* from A to B is a subset $f \subset A \times B$ such that

$$\forall a \in A \exists ! b \in B \ni (a, b) \in f.$$

If f is such a subset of $A \times B$, we indicate this fact by writing $f : A \to B$. If $a \in A$, the unique element of b such that $(a, b) \in f$ is denoted $f(a)$. Functions obey the "defining property":

- for every $a \in A$ there exists $b \in B$ such that $f(a) = b$;
- if $f(a) = b$ and $f(a) = c$, then $b = c$.

Let $f : A \to B$ be a function. The *domain* of f is A, and the *codomain* of f is B.

If $C \subset A$, the *image* of C is $f(C) = \{b \in B \mid f(c) = b \text{ for some } c \in C\}$. The *range* of a function is the image of its domain.

If $D \subset B$, the *preimage* of D is $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$.

We say that f is *injective* (or *one to one*) if for every $a_1, a_2 \in A$ we have $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

We say that f is *surjective* (or *onto*) for every $b \in B$ there exists $a \in A$ such that $f(a) = b$. A function is surjective if and only if its range is equal to its codomain.

We say that f is *bijective* if it is both injective and surjective.

If A is a set, define the *identity function* on A to be the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for all $a \in A$. This function is bijective.

If $f : A \to B$ and $g : B \to C$, define the *composition* of f and g to be the function $g \circ f : A \to B$ given by $g \circ f(a) = g(f(a))$.

We say that f is *invertible* if there exists a function $f^{-1} : B \to A$, called the *inverse* of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proposition 1.3. *A function is invertible if and only if it is bijective.*

If f is injective, we define the *inverse* of f to be a function $f^{-1} : f(A) \to A$ by $f^{-1}(y) = x$, where $f(x) = y$. Since an invertible function is bijective, it is injective, and this definition of inverse agrees with our previous one in this case.

If $f : A \to B$ is a function and $C \subset A$, we define a function $f \restriction_C : C \to B$, called the *restriction* of f to C, by $f \restriction_C (c) = f(c)$. If f is injective, then so is $f \restriction_C$.

6. Cardinality

We say that two sets have the same *cardinality* if and only if there is a bijective function between them.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and for $n \in \mathbb{N}$ let $H_n = \{m \in \mathbb{N} \mid m < n\}$. A set X is called *finite* if there exists a surjective function from X to H_n for some $n \in \mathbb{N}$. If there exists a bijective function $X \to H_n$, we say that the cardinality of X is n, and write $|X| = N$.

A set X is called *infinite* if there exists an injective function $\mathbb{N} \to X$.

Proposition 1.4. *A set is infinite if and only if it is not finite.*

Proposition 1.5. *Let A be a finite set and let $f : A \to A$ be a function. Then f is injective if and only if f is surjective.*

Proposition 1.6. *Let A and B be finite sets. Then $|A \times B| = |A| \cdot |B|$.*

CHAPTER 2

Lecture 2 - Collections and Relations

1. Collections

A *collection* is a set whose elements are themselves sets.

Let X be a set. The collection of all subsets of X is called the *power set* of X and is denoted $\mathcal{P}(X)$.

Let $\mathcal{C}$ be a collection of subsets of X; then $\mathcal{C} \subset \mathcal{P}(X)$. Define the *intersection* and *union* of the collection by

- $\cap\mathcal{C} = \{a \in A \mid a \in C \text{ for all } C \in \mathcal{C}\}$
- $\cup\mathcal{C} = \{a \in A \mid a \in C \text{ for some } C \in \mathcal{C}\}$

If $\mathcal{C}$ contains two subsets of X, this definition concurs with our previous definition for the union of two sets.

Let A and B sets. The collection of all functions from A to B is denoted $\mathcal{F}(A, B)$, and is a subset of $\mathcal{P}(A \times B)$.

2. Families

Let A and X be sets. A *family* of subsets of X indexed by A is the image of an injective function $Y : A \to \mathcal{P}(X)$. For each $a \in A$, the set $Y(a)$ may be denoted by Y_a. The family itself may denoted by $\{Y_a \subset X \mid a \in A\}$

Let $\{Y_a \subset X \mid a \in A\}$ be a family of subsets of a set X. The *intersection* and *union* of the family is defined by

- $\cap_{a\in A} Y_a = \{x \in X \mid x \in Y_a \text{ for all } a \in A\}$;
- $\cup_{a\in A} Y_a = \{x \in X \mid x \in Y_a \text{ for some } a \in A\}$;

Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$ be a collection of subsets of X. Then $\mathcal{C}$ is a family of subsets of X, indexed by itself via the identity function. Our definitions of intersection and union of a family of subsets concur with our definitions of intersections and union of a collection of subsets under this correspondence.

3. Cartesian Product of a Family

Let X be a set and let $\mathcal{C} = \{Y_a \subset X \mid a \in A\}$ be a family of subsets of X. Let $Y = \cup_{a \in A} Y_a$.

The *cartesian product* of $\mathcal{C}$ is denoted by $\times\mathcal{C}$ or by $\times_{a \in A} Y_a$ and is defined to be the collection of all functions from A into the union of the family such that each element of a is mapped to an element of Y_a:

$$\times_{a \in A} Y_a = \{f \in \mathcal{F}(A, Y) \mid f(a) \in Y_a\}.$$

We needed to define the cartesian product of two sets in order to define function, which in turn we have used to define the cartesian product of more than two sets. These definitions concur according to the following proposition.

Proposition 2.1. *Let X be a set and let $Y_1, Y_2 \subset X$. Let $A = \{1, 2\}$ and let $Y = Y_1 \cup Y_2$. For any $x_1, x_2 \in X$, define a function $f_{x_1,x_2} : A \to X$ by $f(1) = x_1$ and $f(2) = x_2$. Define a function*

$$\phi : Y_1 \times Y_2 \to \{f \in \mathcal{F}(A, Y) \mid f(a) \in Y_a\} \quad \text{by} \quad \phi(x_1, x_2) = f_{x_1,x_2}.$$

Then ϕ is a bijection.

4. Relations

A *relation* on a set A is a subset of $R \subset A \times A$. If $(a, b) \in R$, we may indicate this by writing aRb; that is $(a, b) \in R \Leftrightarrow aRb$.

A relation is called *reflexive* if aRa for every $a \in A$.

A relation is called *symmetric* if $aRb \Leftrightarrow bRa$ for every $a, b \in A$.

A relation is called *antisymmetric* if aRb and bRa implies $a = b$.

A relation is called *transitive* if aRb and bRc implies aRc.

A relation is called *definite* if either aRb or bRa for every $a, b \in A$.

A *partial order* on A is a relation which is reflexive, antisymmetric, and transitive. A *total order* on A is a definite partial order.

An *equivalence relation* on A is a relation which is reflexive, symmetric, and transitive.

If $\equiv$ is an equivalence relation on A and $a \in A$, the *equivalence class* of a is the set

$$[a]_{\equiv} = \{b \in A \mid a \equiv b\}.$$

A set is *nonempty* if it is not equal to the empty set. Two sets A, B are called *disjoint* if $A \cap B = \varnothing$. A collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A is said to be *pairwise disjoint* if every distinct pair of members of $\mathcal{C}$ are disjoint. A collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A is said to *cover* A if $\cup\mathcal{C} = A$.

A *partition* of A is a collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A such that

- $\varnothing \notin \mathcal{C}$;
- $\cup\mathcal{C} = A$;
- $A, B \in \mathcal{C} \Rightarrow A \cap B = \varnothing$ or $A = B$.

That is, a partition of A is a pairwise disjoint collection of nonempty subsets of A which covers A.

If $\equiv$ is an equivalence relation A, then the collection of equivalence classes under $\equiv$ is a partition of A. If $\mathcal{C}$ is a partition of A, we may define an equivalence relation $\equiv$ on A by $a \equiv b$ if and only if they are in the same subset of the partition.

If $f : A \to X$ is a surjective function, then the relation $\equiv$ on A defined by $a \equiv b \Leftrightarrow f(a) = f(b)$ is an equivalence relation which partitions A into blocks of elements which are sent to the same place by f. There is a natural bijective function from the partition into X given by sending each block to the appropriate element in X.

CHAPTER 3

Lecture 3 - Induction

1. Set Proof Example

The following properties are sometimes useful in proofs:

- $A = A \cup A = A \cap A$
- $\varnothing \cap A = \varnothing$
- $\varnothing \cup A = A$
- $A \subset B \Leftrightarrow A \cap B = A$
- $A \subset B \Leftrightarrow A \cup B = B$

As an example, we prove one of these properties.

Proposition 3.1. *Let A and B be a sets. Then $A \subset B \Leftrightarrow A \cap B = A$.*

Proof. To prove an if and only if statement, we prove implication in both directions.

($\Rightarrow$) Assume that $A \subset B$. We wish to show that $A \cap B = A$. To show that two sets are equal, we show that each is contained in the other.

($\subset$) To show that $A \cap B \subset A$, it suffices to show that every element of $A \cap B$ is in A. Thus we select an arbitrary element $c \in A \cap B$ and show that it is in A. Now by definition of intersection, $c \in A \cap B$ means that $c \in A$ and $c \in B$. Thus $c \in A$. Since c was arbitrary, every element of $A \cap B$ is contained in A. Thus $A \cap B \subset A$.

($\supset$) Let $a \in A$. We wish to show that $a \in A \cap B$. Since $A \subset B$, then every element of A is an element of B. Thus $a \in B$. So $a \in A$ and $a \in B$. By definition of intersection, $a \in A \cap B$. Thus $A \subset A \cap B$.

Since $A \cap B \subset A$ and $A \subset A \cap B$, we have $A \cap B = A$.

($\Leftarrow$) Assume that $A \cap B = A$. We wish to show that $A \subset B$. Let $a \in A$. It suffices to show that $a \in B$. Since $A \cap B = A$, then $a \in A \cap B$. Thus $a \in A$ and $a \in B$. In particular, $a \in B$. □

Now let us prove the analogous statement in compressed form.

Proposition 3.2. *Let A and B be a sets. Then $A \subset B \Leftrightarrow A \cup B = B$.*

Proof.

($\Rightarrow$) Assume that $A \subset B$. Clearly $B \subset A \cup B$, so we show that $A \cup B \subset B$. Let $c \in A \cup B$. Then $c \in A$ or $c \in B$. If $c \in B$ we are done, so assume that $c \in A$. Since $A \subset B$, then $c \in B$ by definition of subset. Thus $A \cup B \subset B$.

($\Leftarrow$) Assume that $A \cup B = B$ and let $a \in A$. Thus $a \in A \cup B$, so $a \in B$. Thus $A \subset B$. □

2. Natural Numbers

Define the natural numbers.

- $0 = \varnothing$;

- $1 = \{\varnothing\}$;
- $2 = \{\varnothing, \{\varnothing\}\}$;
- $3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$;

and so forth. We could have written this as

- $0 = \varnothing$;
- $1 = \{0\}$;
- $2 = \{0, 1\}$;
- $3 = \{0, 1, 2\}$;

and so forth. A given natural number is the set containing all of the previous natural numbers. Restate as follows.

We define 0 to be the empty set. If x is a set, the *successor* of x is denoted x^+ and is defined as

$$x^+ = x \cup \{x\}.$$

The *natural numbers* are the set $\mathbb{N}$ defined by following properties:

(1) $0 \in \mathbb{N}$;
(2) if $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$;
(3) if $S \subset \mathbb{N}$, $0 \in S$, and $n \in S \Rightarrow n^+ \in S$, then $S = \mathbb{N}$.

For $m, n \in \mathbb{N}$, we say the m is less than or equal to n if $m \subset n$:

$$m \leq n \Leftrightarrow m \subset n.$$

3. Induction

Note that the third property of natural numbers asserts that only successors of 0 are in $\mathbb{N}$; that is, this property asserts that $\mathbb{N}$ is a minimal set of successors of 0, and that $\mathbb{N}$ is the unique set satisfying (1) through (3). This property is known as the *Principal of Mathematical Induction*.

Suppose that for every natural number n, we have a proposition $p(n)$ which is either true or false. Let

$$S = \{n \in \mathbb{N} \mid p(n) \text{ is true}\}.$$

Now if $p(0)$ is true, and if the truth of $p(n)$ implies the truth of $p(n^+)$, then the set S contains 0 and it contains the successor of every element in it. Thus, in this case, $S = \mathbb{N}$, which means that $p(n)$ is true for all $n \in \mathbb{N}$. We state this as

Theorem 3.3. Induction Theorem
Let $p(n)$ be a proposition for each $n \in \mathbb{N}$. If

(1) *$p(0)$ is true;*
(2) *If $p(n)$ is true, then $p(n^+)$ is true;*

then $p(n)$ is true for all $n \in \mathbb{N}$.

Example 3.4. Show that $\sum_{i=1}^{n} = \frac{(n-1)n}{2}$ for all $n \in \mathbb{N}$.

Example 3.5. Show that $7 \mid (11^n - 4^n)$ for all $n \in \mathbb{N}$.

Proof. For $n = 1$, we have $7 = 11 - 4$, so clearly $7 \mid 11^1 - 4^1$. Thus assume that $7 \mid 11^{n-1} - 4^{n-1}$, so there exists $x \in \mathbb{Z}$ such that $7x = (11^{n-1} - 4^{n-1})$. Now

$$\begin{aligned} 11^n - 4^n &= 11^n - 11 \cdot 4^{n-1} + 11 \cdot 4^{n-1} - 4 \cdot 4^{n-1} \\ &= (11^{n-1} - 4^{n-1})11 + (11 - 4)4^{n-1} \\ &= 7x \cdot 11 + 7 \cdot 4^{n-1} \\ &= 7(11x + 4^{n-1}. \end{aligned}$$

Thus $7 \mid (11^n - 4^n)$. □

Now the induction theorem can be made stronger by weakening the hypothesis. The resulting theorem gives a proof technique which is known as strong induction.

Theorem 3.6. Strong Induction Theorem
Let $p(n)$ be a proposition for each $n \in \mathbb{N}$. If

(1) *$p(0)$ is true;*
(2) *If $p(m)$ is true for all $m \leq n$, then $p(n + 1)$ is true;*

then $p(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $t(n)$ be the statement that "p(m) is true for all $m \leq n$".

Our first assumption is that $p(0)$ is true, and since the only natural number less than or equal to 0 is zero (because the only subset of the empty set is itself), this means that $t(0)$ is true.

Our second assumption is that if $t(n)$ is true, then $p(n+1)$ is true. Thus assume that $t(n)$ is true so that $p(n + 1)$ is also true. Then $p(i)$ is true for all $i \leq n + 1$. Thus $t(n + 1)$ is true.

By our original Induction Theorem, we conclude that $t(n)$ is true for all $n \in \mathbb{N}$. This implies that $p(n)$ is true for all $n \in \mathbb{N}$. □

4. Recursion

We now state the Recursion Theorem, which will allows us to define addition and multiplication of natural numbers.

Theorem 3.7. Recursion Theorem
Let X be a set, $f : X \to X$, and $a \in X$. Then there exists a unique function $\phi : \mathbb{N} \to X$ such that $\phi(0) = a$ and $\phi(n^+) = f(\phi(n))$ for all $n \in \mathbb{N}$.

Reason. May be proved by induction. □

Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = n^+$. Let $\sigma_m : \mathbb{N} \to \mathbb{N}$ be the unique function, whose existence is guaranteed by the Recursion Theorem, defined by $\sigma_m(0) = m$ and $\sigma_m(n^+) = f(\sigma_m(n)) = (\sigma_m(n))^+$. Then $\sigma_m(n)$ is defined to be the *sum* of m and n:

$$m + n = \sigma_m(n).$$

Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f = \sigma_m$. Let $\mu_m : \mathbb{N} \to \mathbb{N}$ be the unique function, whose existence is guaranteed by the Recursion Theorem, defined by $\mu_m(0) = 0$ and $\mu_m(n^+) = f(\mu_m(n)) = \sigma_m(\mu_m(n)) = m + \mu_m(n)$. Then $\mu_m(n)$ is defined to be the *product* of m and n:

$$mn = \mu_m(n).$$

The following properties of natural numbers can be proved using the above definitions:

- $m + n = n + m$ (commutativity of addition);
- $(m + n) + o = m + (n + o)$ (associativity of addition);
- $mn = nm$ (commutativity of multiplication);
- $(mn)o = m(no)$ (associativity of multiplication);
- $m(n + o) = mn + mo$ (distributivity of multiplication over addition);
- $m + 0 = m$ (0 is an additive identity);
- $1m = m$ (1 is a multiplicative identity);
- $0m = 0$.

We state two additional properties, which we will use to show that multiplication of integers is well-defined.

Proposition 3.8. Cancellation Law of Addition
Let $a, b, c \in \mathbb{N}$ and suppose that $a + c = b + c$. Then $a = b$.

Proposition 3.9. Cancellation Law of Multiplication
Let $a, b, c \in \mathbb{N}$ and suppose that $ac = bc$. Then $a = b$.

5. Integers

Develop the integers from the natural numbers as follows.

Let $A = \mathbb{N} \times \mathbb{N}$. We wish to think of the elements (a, b) of A as differences $a - b$.

Define a relation $\sim$ on A by

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c.$$

Prove that this is an equivalence relation. Let $[a, b]$ denote the equivalence class of (a, b).

Set $\mathbb{Z} = \{[a, b] \mid a, b \in \mathbb{N}\}$.

Define addition and multiplication on $\mathbb{Z}$ as follows:

- $[a, b] + [c, d] = [a + c, b + d]$;
- $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$.

Prove that these binary operations are well-defined and satisfy the desired properties of the integers. The additive identity is $[0, 0]$ and the additive inverse of $[a, b]$ is $[b, a]$. The multiplicative identity is $[1, 0]$.

Define a relation $\leq$ on $\mathbb{Z}$ by

$$[a, b] \leq [c, d] \Leftrightarrow a + d \leq b + c.$$

Prove that this is a linear order relation on $\mathbb{Z}$, and that it relates to addition and multiplication in the desired way.

6. Rationals

Develop the rationals from the integers as follows.

Let $A = \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. We wish to think of the elements (a, b) of A as fractions $\frac{a}{b}$.

Define a relation $\sim$ on A by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Prove that this is an equivalence relation. Let $[a, b]$ denote the equivalence class of (a, b).

Set $\mathbb{Q} = \{[a, b] \mid a, b \in \mathbb{Z} \text{ with } b \neq 0\}$.

Define addition and multiplication on $\mathbb{Q}$ as follows:

- $[a, b] + [c, d] = [ad + bc, bd]$;
- $[a, b] \cdot [c, d] = [ac, bd]$.

Prove that these binary operations are well-defined and satisfy the desired properties of the integers. The additive identity is $[0, 1]$ and the additive inverse of $[a, b]$ is $[-a, b]$. The multiplicative identity/is $[1, 1]$ and the multiplicative inverse of $[a, b]$ is $[b, a]$. Denote $[0, 1]$ by 0 and $[1, 1]$ by 1. For $x = [a, b]$, denote $[-a, b]$ by $-x$ and $[b, a]$ by x^{-1}.

Define a relation $\leq$ on $\mathbb{Q}$ by

$$[a, b] \leq [c, d] \Leftrightarrow (ad - bc)bd \leq 0.$$

Prove that this is a linear order relation on $\mathbb{Q}$, and that it relates to addition and multiplication in the desired way.

The set $\mathbb{Q}$ satisfies the following properties:

(F1) $(x + y) + z = x + (y + z)$;
(F2) $x + 0 = x$;
(F3) $x + (-x) = 0$;
(F4) $xy = yx$;
(F5) $(xy)z = x(yz)$;
(F6) $x \cdot 1 = x$;
(F7) $x \cdot x^{-1} = 1$;
(F8) $xy = yx$;
(F9) $x(y + z) = xy + xz$;
(O1) $x \leq x$;
(O2) $x \leq y$ and $y \leq x$ implies $x = y$;
(O3) $x \leq y$ and $y \leq z$ implies $x \leq z$;
(O4) $x \leq y$ or $y \leq x$.

Properties **(F1)** through **(F2)** say that $\mathbb{Q}$ is a *field*, and properties **(O1)** through **(O4)** say that $\mathbb{Q}$ is a *linearly ordered set.*

CHAPTER 4

Lecture 4 - Dedekind Cuts

1. Example of Strong Induction

Let $x \in \mathbb{Z}$, $x \geq 2$. We say that x is *prime* if whenever $x = ab$ for some positive integers a and b, then either $a = 1$ or $b = 1$.

Problem 4.1. Let $x \in \mathbb{Z}$ be a positive integer, $x \geq 2$. Then x is the product of prime integers.

Proof. Proceed by induction on x, and select $x = 2$ as the base case. Clearly 2 is prime, and so it is the product of primes.

Now assume that every integer between 2 and $x - 1$ is a product of prime integers. If x is itself prime, we are done, so assume that x is not prime. Then there exist $a, b \in \mathbb{Z}$ such that $x = ab$ with $a \neq 1$ and $b \neq 1$. Then $a < x$ and $b < x$, so a is the product of primes and b is the product of primes. Therefore x is the product of primes. □

2. Jargon

Maximal and minimal (extremal) versus maximum and minimum (extremum).

Supremal and infimal versus supremum and infimum.

3. Linearly Ordered Sets

A *linearly ordered set* $(A, \leq)$ is a set A together with a relation $\leq$ on A satisfying

(O1) $a \leq a$;
(O2) $a \leq b$ and $b \leq a$ implies $a = b$;
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$;
(O4) $a \leq b$ or $b \leq a$;

for every $a, b, c \in A$. We call $\leq$ a *linear order relation* on A.

Let A and B be linearly ordered sets. A *morphism* from A to B is a function $f : A \to B$ such that

$$a_1 \leq a_2 \Rightarrow f(a_1) \leq f(a_2).$$

Let A be a linearly ordered set. If $B \subset A$, the B naturally inherits the linear order, and becomes a linearly ordered set in its own right.

Let $b \in B$. We say that b is an *minimal element* of B if $b \leq c$ for every $c \in B$. Similarly, we say that b is a *maximal element* of B if $c \leq b$ for every $c \in B$.

We say that A is *dense* if for every $a_1, a_2 \in A$ with $a_1 < a_2$, there exists $a \in A$ such that $a_1 < a < a_2$.

Consider a partition $\{C, U\}$ of A into two blocks such that $c \leq u$ for every $c \in C$ and $u \in U$. There are four possibilities:

(a) C has a maximal element and U has a minimal element;
(b) C has a maximal element and U does not have a minimal element;
(c) C does not have a maximal element and U has a minimal element;
(d) C does not have a maximal element and U does not have a minimal element.

In cases **(a)**, **(c)**, and **(d)**, we say that C is a *cut*.
In case **(a)**, we say that C is a *jump*.
In case **(c)**, we say that C is a *hit*.
In case **(d)**, we say that C is a *gap*.

Observation 1. Let A be a linearly ordered set. Then A is dense if and only if A has no jumps.

Observation 2. The rational numbers $\mathbb{Q}$ is dense.

Proof. The average of two distinct rational numbers is rational and is between them. $\square$

4. Dedekind Cuts

A *Dedekind cut* is a cut in the rational number line; that is, it is a proper nonempty subset $C \subset \mathbb{Q}$ such that

(C1) $c \in C$ and $u \in \mathbb{Q} \smallsetminus C$ implies $c < u$;
(C2) C has does not contain a maximal element.

The set of all Dedekind cuts is naturally ordered by inclusion. Moreover, this is a total order

5. Addition of Cuts

Let C_1 and C_2 be Dedekind cuts. Define their sum as

$$C_1 + C_2 = \{x \in \mathbb{Q} \mid x = c_1 + c_2 \text{ for some } c_1 \in C_1 \text{ and } c_2 \in C_2\}.$$

Proposition 4.1. *Let $C_1, C_2 \subset \mathbb{Q}$ be cuts. Then $C_1 + C_2$ is a cut.*

Proof. Set $C = C_1 + C_2$ and $U = \mathbb{Q} \smallsetminus C$. Clearly $C \subset \mathbb{Q}$ is nonempty; we wish to prove properties **(C1)** and **(C2)**.

Let $c \in C$ and $u \in U$. Then $c = c_1 + c_2$ for some $c_1 \in C_1$ and $c_2 \in C_2$. Suppose that $u \leq c$; then $u - c_2 \leq c_1$, which implies that $u - c_2 \in C_1$. Set $u - c_2 = a \in C_1$; then $u = a + c_2 \in C_1 + C_2 = C$, a contradiction. Thus $c < u$.

Since C_1 and C_2 are cuts, c_1 and c_2 are not maximal elements in C_1 and C_2, respectively. Thus there exists $a_1 \in C_1$ and $a_2 \in C_2$ such that $c_1 < a_1$ and $c_2 < a_2$. Then $a_1 + a_2 \in C$, and $c < a_1 + a_2$; thus c is not maximal in C. □

Let $M = \{x \in \mathbb{Q} \mid x < 0\}$. Clearly M is a Dedekind cut.

Let C be a Dedekind cut, and set

$$-C = \{x \in \mathbb{Q} \mid x = -y \text{ for some nonminimal } y \in \mathbb{Q} \smallsetminus C\}.$$

Proposition 4.2. *Let $C, C_1, C_2, C_3 \subset \mathbb{Q}$ be cuts. Then $-C$ is a Dedekind cut, and*

(F1) $(C_1 + C_2) + C_3 = C_1 + (C_2 + C_3)$;
(F2) $C + M = C$;
(F3) $C + (-C) = M$;
(F4) $C_1 + C_2 = C_2 + C_1$.

Proof. Exercise.

□

Define subtraction of Dedekind cuts in the usual way.

Let C be a Dedekind cut, and say that C is *positive* if M is a proper subset of C, and that C is *negative* if C is a proper subset of M.

6. Multiplication of Cuts

Let C_1 and C_2 be Dedekind cuts. Set

$$C_1 * C_2 = \{x \in \mathbb{Q} \mid x = c_1 c_2 \text{ for some } c_1 \in C_1 \setminus M \text{ and } c_2 \in C_2 \setminus M\} \cup M.$$

Now define their product by

$$C_1 \cdot C_2 = \begin{cases} C_1 * C_2 & \text{if } C_1 \text{ and } C_2 \text{ are positive;} \\ -((-C_1) * C_2) & \text{if } C_1 \text{ is negative and } C_2 \text{ is positive;} \\ -(C_1 * (-C_2)) & \text{if } C_1 \text{ is positive and } C_2 \text{ is negative;} \\ (-C_1) * (-C_2) & \text{if } C_1 \text{ and } C_2 \text{ are negative;} \\ M & \text{if } C_1 = M \text{ for } C_2 = M. \end{cases}$$

Proposition 4.3. *Let $C_1, C_2 \subset \mathbb{Q}$ be cuts. Then $C_1 \cdot C_2$ is a cut.*

Proof. Again set $C = C_1 + C_2$ and $U = \mathbb{Q} \setminus C$, and prove properties **(C1)** and **(C2)**. We assume that C_1 and C_2 are positive; the other cases require only minor adjustments.

Let $c \in C$ so that $c = c_1 c_2$ for some $c_1 \in C_1 \setminus M$ and $c_2 \in C_2 \setminus M$; the other cases are easy.

Let $u \in U$; by definition, $M \subset C$ so $0 \leq u$. Suppose that $u \leq c$; then $u/c_2 \leq c_1$, which implies that $u/c_2 \in C_1$. Set $u/c_2 = a \in C_1$; then $u = ac_2 \in C_1 \cdot C_2 = C$, a contradiction. Thus $c < u$.

Since C_1 and C_2 are cuts, c_1 and c_2 are not maximal elements in C_1 and C_2, respectively. Thus there exists $a_1 \in C_1$ and $a_2 \in C_2$ such that $c_1 < a_1$ and $c_2 < a_2$. Then $a_1 a_2 \in C$, and $c < a_1 a_2$; thus c is not maximal in C. □

Let $I = \{x \in \mathbb{Q} \mid x < 1\}$. Clearly I is a Dedekind cut.

Let C be a Dedekind cut different from M, and set

$$C^{-1} = \{x \in \mathbb{Q} \mid x = y^{-1} \text{ for some } y \in C\}.$$

Proposition 4.4. *Let $C, C_1, C_2, C_3 \subset \mathbb{Q}$ be cuts. Then C^{-1} is a Dedekind cut, and*

- **(F5)** $(C_1 \cdot C_2) \cdot C_3 = C_1 \cdot (C_2 \cdot C_3)$;
- **(F6)** $C \cdot I = C$;
- **(F7)** $C \cdot (C^{-1}) = I$;
- **(F8)** $C_1 \cdot C_2 = C_2 \cdot C_1$;
- **(F9)** $C_1 \cdot (C_2 + C_3) = (C_1 \cdot C_2) + (C_1 \cdot C_3)$.

Proof. Exercise. □

7. Ordering of Cuts

Let $\mathcal{C} = \{C \subset \mathbb{Q} \mid C \text{ is a cut}\}$. Define a relation $\leq$ on $\mathcal{C}$ by

$$C_1 \leq C_2 \Leftrightarrow C_1 \subset C_2.$$

Proposition 4.5. *Let $C, C_1, C_2, C_3 \in \mathcal{C}$. Then*

(O1) $C \leq C$;
(O2) $C_1 \leq C_2$ *and* $C_2 \leq C_1$ *implies* $C_1 = C_2$;
(O3) $C_1 \leq C_2$ *and* $C_2 \leq C_3$ *implies* $C_1 \leq C_3$;
(O4) $C_1 \leq C_2$ *or* $C_2 \leq C_1$.

Moreover,

(O5) $C_1 \leq C_2$ *implies* $C_1 + C_3 \leq C_2 + C_3$;
(O6) $C_1 \leq C_2$ *implies* $C_1 \cdot C_3 \leq C_2 \leq C_3$ *whenever* $M \leq C_3$.

Proof. Exercise. □

CHAPTER 5

Lecture 5 - Suprema and Infima

1. The Real Numbers

Define the *real numbers* to be the set of all Dedekind cuts, and denote this set by $\mathbb{R}$. This is an ordered field.

For every $a \in \mathbb{Q}$, let $C_a = \{x \in \mathbb{Q} \mid x < a\}$. This is clearly a Dedekind cut; we call this the *rational cut* corresponding to a. Note that a cut is rational if and only if it is a hit.

Define a function

$$\phi : \mathbb{Q} \to \mathbb{R} \quad \text{by} \quad \phi(a) = C_a.$$

This function satisfies the following properties:

(H1) $\phi(1) = I$ (where I is the multiplicative identity in $\mathbb{R}$);
(H2) $\phi(a + b) = \phi(a) + \phi(b)$;
(H3) $\phi(ab) = \phi(a)\phi(b)$.

These properties say that ϕ is a *field homomorphism*. The image $\phi(\mathbb{Q})$ is a subfield of $\mathbb{R}$ which is isomorphic to $\mathbb{Q}$ as an ordered field. Thus we may identify the rational numbers with the set of rational cuts, and we no longer make a distinction between them. We now view $\mathbb{Q}$ as a subset of $\mathbb{R}$.

Next we note that this process has produced real numbers which did not exist in $\mathbb{Q}$; that is, the function ϕ is not surjective. To see this, we use the following exercise:

Problem 5.1. Let $a, b \in \mathbb{Q}$ with $0 < a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q^2 < b$.

From the rational roots theorem, we know that there is no rational number whose square is 2. However, there is a real number with this property.

Example 5.1. Set $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Then $C \cdot C = \{x \in \mathbb{Q} \mid x < 2\} = C_2$.

Proof. First note that $1 \in C$, implying that C is a positive cut. Recall that

$$C \cdot C = \{x \in \mathbb{Q} \mid x = ab \text{ with } a, b \in C \setminus M\} \cup M.$$

Let $x \in C \cdot C$. If $x \leq 0$, then $x \in C_2$, so assume that $x > 0$. Then $x = ab$ for some $a, b \in C \setminus M$. Without loss of generality, assume that $b > a$. Then $x = ab \leq b^2 < 2$, so $x \in C_2$.

Let $x \in C_2$, so that $x < 2$. By the previous problem, there exists $q \in \mathbb{Q}$ such that $x < q^2 < 2$. Thus $q \in C$ so $q^2 \in C \cdot C$; since $C \cdot C$ is a cut and $x < q^2$, we must have $x \in C \cdot C$. □

Every Dedekind cut is either a hit or a gap. The image of ϕ is exactly the set of hits in the rational number line, and the irrational numbers is exactly the set of gaps.

Finally, we ask if it is possible to produce even more numbers if we repeat this process; that is, does the set of real numbers have any gaps? We demonstrate this it does not, and explore the consequences of this property.

First we need to recall a basic property of sets.

Proposition 5.2 (DeMorgan's Laws for Sets)**.** *Let W be a set. Let S be a subset of W and let $\mathcal{C}$ be a collection of subsets of W. Then*

- **(a)** $S \smallsetminus \cup_{C\in\mathcal{C}} C = \cap_{C\in\mathcal{C}}(S \smallsetminus C)$*;*
- **(b)** $S \smallsetminus \cap_{C\in\mathcal{C}} C = \cup_{C\in\mathcal{C}}(S \smallsetminus C)$.

Theorem 5.3 (Cantor-Dedekind Theorem)**.** *The set of real numbers has no gaps.*

Proof. Let $\mathcal{C}$ be a cut in $\mathbb{R}$. Let $A = \cup_{C\in\mathcal{C}} C$. Set $U = \mathbb{Q} \smallsetminus A$. By DeMorgan's Law, $U = \cap_{C\in\mathcal{C}} C$.

Claim 1: A is a cut.

Let $a \in A$ and $u \in U$; we wish to show that $a < c$. Then $u \in \mathbb{Q} \smallsetminus C$, for every $C \in \mathcal{C}$. But $a \in C$ for some $C \in \mathcal{C}$, and u is not in C; since C is a cut, $a < u$.

Let $a \in A$; we wish to show that a is not maximal in A. Now $a \in C$ for some $C \in \mathcal{C}$, and since C is a cut, a is not maximal in C, so there exists $c \in C$ such that $a < c$. But $C \subset A$, so $c \in A$ and a is not maximal in A.

Claim 2: $A \in \mathbb{R} \smallsetminus \mathcal{C}$

If A were in $\mathcal{C}$, it would be the largest element in $\mathcal{C}$, because the ordering is inclusion and A contains every set in $\mathcal{C}$. In this case $\mathcal{C}$ would not be a cut.

Claim 3: A is minimal in $\mathbb{R} \smallsetminus \mathcal{C}$

Let X be a cut, and suppose that $X < A$; we wish to show that X is in $\mathcal{C}$. Now $X < A$ means that X is strictly contained in A, so there exists $y \in A$ such that $y \notin X$. However, since $y \in A$, we know that $y \in Y$ for some $Y \in \mathcal{C}$, and we have $X < Y < A$. Since $\mathcal{C}$ is a cut, $X \in \mathcal{C}$.

This completes the proof that $\mathcal{C}$ is not a gap. □

2. Suprema and Infima

Let A be a linearly ordered set, and let $B \subset A$.

An *upper bound* for B is an element $a \in A$ such that $b \leq a$ for every $b \in B$. If B has an upper bound, we say that B is *bounded above*.

A *lower bound* for B is an element $a \in A$ such that $a \leq b$ for every $b \in B$. If B has a lower bound, we say that B is *bounded below*.

We say that B is *bounded* if it is both bounded above and bounded below.

A *supremum* of B is an element $a \in A$ such that

- **(a)** $b \leq a$ for every $b \in B$;
- **(b)** $b \leq c$ for every $b \in B$ implies $b \leq c$.

In this case, write $a = \sup(B)$.

An *infimum* of B is an element $a \in A$ such that

- **(a)** $b \geq a$ for every $b \in B$;
- **(b)** $b \geq c$ for every $b \in B$ implies $b \geq c$.

In this case write $a = \inf(B)$.

Suprema and infima are unique, if they exist. For this reason, they are sometimes referred to as *least upper bound* (lub) and *greatest lower bound* (glb), respectively.

Observation 3. Let A be a linearly ordered set. The following are equivalent conditions on A:

- **(a)** every nonempty subset of A that is bounded above has an lub;
- **(b)** every nonempty subset of A that is bounded below has a glb;
- **(c)** A has no gaps.

A linearly ordered set satisfying any one of these equivalent conditions is called *complete*.

Proposition 5.4. *The real numbers are a complete ordered field.*

Henceforth, we prove all that we need using this characterization of the real numbers.

3. Density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$

Let $\mathbb{R}$ be the set of real numbers. View the $\mathbb{Q}$ denote the set of rationals cuts, which we now refer to as rational numbers. Set $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$; this is the set of *irrational numbers.*

Proposition 5.5 (Archimedean Property)**.** *Let $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$. Then there exists $n \in \mathbb{N}$ such that $b \leq na$.*

Proof. Suppose that the Archimedean property fails. Then there exists $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$ such that $na < b$ for every $n \in \mathbb{N}$. Set $A = \{na \mid n \in \mathbb{N}\}$. Now A is bounded above by b, so by the completeness property of $\mathbb{R}$, there exists $s \in \mathbb{R}$ such that $s = \sup(A)$. Since $a > 0$ we have $s < s + a$, so $s - a < s$. Since s is a least upper bound for A, $s - a$ is not an upper bound for A; thus there exists $na \in A$ such that $s - a < na$. This implies that $s < (n+1)a \in A$, which contradicts that $s = \sup(A)$. □

Proposition 5.6 (Density of $\mathbb{Q}$)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. Set $c = b - a$, and note that $c > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $1 \leq nc$, which shows that $1 \leq nb - na$, or $na + 1 \leq nb$.

Let $m = \min\{x \in \mathbb{N} \mid na < x\}$; this m exists by the Well-Ordering Principle of the natural numbers. Now $m \leq na + 1$, for otherwise $na + 1 < m$ and $na < m - 1 < m$, contradicting the minimality of m. Therefore $na < m < na + 1 < nb$. Divide by n to achieve $a < \frac{m}{n} < b$. With $q = \frac{m}{n}$, the proof is complete. □

Proposition 5.7 (Density of $\mathbb{I}$)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists $x \in \mathbb{I}$ such that $a < x < b$.*

Proof. First observe that if $q \in \mathbb{Q}$ and $x \in \mathbb{I}$, then $q + x \in \mathbb{I}$.

Let $q \in \mathbb{Q}$ be a rational number such that $a - x < q < b - x$. Then $a < q + x < b$, with $q + x \in \mathbb{I}$. □

CHAPTER 6

Lecture 6 - Cardinality

1. Motivation

We have seen that there are infinitely many rational numbers, and infinitely many irrational numbers. So the question arises as to whether or not there are as many rational numbers as there are real numbers: there are infinitely many of both. We know that the rational numbers embed into the real numbers, but does there exist an injective function in the other direction?

We begin by demonstrating that there is more than one type of infinite set in this regard.

Proposition 6.1. *Let X be a set. Then there does not exist a surjective function $X \to \mathcal{P}(X)$.*

Proof. Let $f : X \to \mathcal{P}(X)$; we wish to show that f is not surjective. Set

$$Y = \{x \in X \mid x \notin f(x)\}.$$

Suppose, by way of contradiction, that $f(x) = Y$ for some $x \in X$. Is $x \in Y$? If it is, then $x \in f(x)$, so by definition of Y, $x \notin Y$. On the other hand, if it is not, then $x \notin f(x)$, so $x \in Y$. Either case is an immediate contradiction. Thus there is no such x satisfying $f(x) = Y$, and Y is not in the image of f. Therefore f is not surjective. □

2. Cardinal Numbers

Let U be a set; we refer to U as a *universal set*, and assume that U contains $\mathbb{R}$.

Let A and B be sets. We say that A and B have the same *cardinality* if there exists a bijective function between them. If A and B have the same cardinality, we write $A \sim B$. Then $\sim$ is a relation on $\mathcal{P}(U)$.

Proposition 6.2. *The relation $\sim$ is an equivalence relation on $\mathcal{P}(U)$.*

We shall call the equivalence classes of the relation the *cardinal numbers in U*. Let $\gimel$ denote the set of cardinal numbers in U. If $A \subset U$, the equivalence class to which it belongs is denoted $|A|$, and is called the *cardinality* of A.

Define a relation $\leq$ on $\gimel$ by

$$|A| \leq |B| \Leftrightarrow \exists \text{ injective } f : A \to B;$$

where $A, B \subset U$ are representatives of the cardinal numbers $|A|$ and $|B|$ respectively.

Proposition 6.3. *The relation $\leq$ on $\gimel$ is well defined.*

That is, let $A_1, A_2, B_1, B_2 \subset U$ such that $A_1 \sim A_2$ and $B_1 \sim B_2$, and such that $|A_1| \leq |B_1|$. Show that $|A_2| \leq |B_2|$.

3. Schroeder-Bernstein Theorem

Lemma 6.4 (Banach's Lemma)**.** *Let X and Y be sets. and let $f : X \to Y$ and $g : Y \to X$ be injective functions. There exist subsets $A \subset X$ and $B \subset Y$ such that $f(A) = B$ and $g(Y \setminus B) = X \setminus A$.*

Proof. Fix the following objects:

- Let X and Y be sets.
- Let $f : X \to Y$ and $g : Y \to X$ be injective functions.
- Let $h = g \circ f$.
- Let $C_0 = X \setminus g(Y)$.
- Let $C_n = h(C_{n-1})$, for each $n \in \mathbb{N}$.
- Let $A = \cup_{n=0}^{\infty} C_n$.
- Let $B = f(A)$.

It suffices to show that $g(Y \setminus B) = X \setminus A$.

Claim 1: $h(A) \subset A$.
Let $a_0 \in h(A)$. Then $a_0 = h(a_1)$ for some $a_1 \in A$. By definition of A, $a_1 \in C_n$ for some $n \in \mathbb{N}$. Then $a_0 \in C_{n+1}$. Thus $a_0 \in A$.

Claim 2: $g(Y \setminus B) \subset X \setminus A$.
We want to select an arbitrary $y_0 \in Y \setminus B$ and show that g sends it into $X \setminus A$. Let $x_0 \in g(Y \setminus B)$. Then there exists $y_0 \in Y \setminus B$ such that $g(y_0) = x_0$. Suppose by way of contradiction that $x_0 \in A$. Since $x_0 \in g(Y)$, $x_0 \notin C_0$, so $x_0 \in C_n$ for some $n > 0$. Since $C_n = h(C_{n-1})$, there exists $x_1 \in C_{n-1}$ such that $h(x_1) = x_0$. So $g(f(x_1)) = x_0$. Since g is injective, $f(x_1) = y_0$. But $x_1 \in A$, so $y_0 \in B$. This is a contradiction. Thus $x_0 \notin A$, so $x_0 \in X \setminus A$. Since x_0 was chosen arbitrarily, $g(Y \setminus B) \subset X \setminus A$.

Claim 3: $g(Y \setminus B) \supset X \setminus A$.
We want to select an arbitrary $x_0 \in X \setminus A$ and find $y_0 \in Y \setminus B$ which g sends to it. Let $x_0 \in X \setminus A$. Since $C_0 \subset A$, then $x_0 \in X \setminus C_0$. That is, $x_0 \in g(Y)$, so there exists $y_0 \in Y$ such that $g(y_0) = x_0$. Suppose by way of contradiction that $y_0 \in B$. Then there exists $x_1 \in A$ such that $f(x_1) = y_0$. Thus $h(x_1) = x_0$, so $x_0 \in h(A)$. Since $h(A) \subset A$, $x_0 \in A$, which is a contradiction. Thus $y_0 \notin B$, so $x_0 \in g(Y \setminus B)$. Since x_0 was chosen arbitrarily, $X \setminus A \subset g(Y \setminus B)$. □

Theorem 6.5 (The Schroeder-Bernstein Theorem)**.** *Let X and Y be sets. If there exist injective functions $f : X \to Y$ and $g : Y \to X$, then there exists a bijective function $h : X \to Y$.*

Proof. Let A and B be sets as specified by the lemma. Let $V = X \setminus A$ and $W = Y \setminus B$. Then $f \restriction_A : A \to B$ is bijective, and $g \restriction_W : W \to V$ is bijective. Let $r = (g \restriction_W)^{-1}$. Then $r : V \to W$ is bijective. Thus define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ r(x) & \text{if } x \in V. \end{cases}$$

□

4. Axiom of Choice

Assume the following version of a famous axiom from set theory.

Axiom 6.6. Axiom of Choice
Let A and B be sets.

(a) *There exists either a surjective function $A \to B$ or a surjective function $B \to A$.*
(b) *There exists an injective function $f : A \to B$ if and only if there exists a surjective function $g : B \to A$.*

Corollary 6.7. *Let X and Y be sets. If there exist surjective functions $f : X \to Y$ and $g : Y \to X$, then there exists an bijective function $h : X \to Y$.*

Proof. This follows immediately by combining the Schroeder-Bernstein Theorem with the Axiom of Choice. □

Corollary 6.8. *Let X and Y be sets. The following conditions are equivalent:*

- $|X| = |Y|$;
- $\exists$ *a bijective function $X \to Y$;*
- $\exists$ *injective functions $X \to Y$ and $Y \to X$;*
- $\exists$ *surjective functions $X \to Y$ and $Y \to X$.*

Proposition 6.9. *Show that $(\beth, \leq)$ is an ordered set.*

Proof. To show this, one must show that $\leq$ is a total order relation on $\mathcal{P}(U)$. The proof of symmetry uses the Schroeder Bernstein Theorem, and the proof of definiteness requires the Axiom of Choice. □

The total order relation $\leq$ on $\beth$ naturally leads to the following definitions for derived relations on $\beth$:

- $|A| \geq |B| \Leftrightarrow |B| \leq |A|$;
- $|A| < |B| \Leftrightarrow \neg(|A| \geq |B|)$;
- $|A| > |B| \Leftrightarrow \neg(|A| \leq |B|)$.

CHAPTER 7

Lecture 7 - Countability and Uncountability

1. Countability

Let $n \in \mathbb{N}$ and set $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$. By convention, we write that $|\mathbb{N}_n| = n$.

Let A be a set. We say that A is *finite* if $|A| = n$ for some $n \in \mathbb{N}$, that is, if there exists a bijective function $A \to \mathbb{N}_n$. We say that A is *infinite* if it is not finite.

We say that A is *countable* if $|A| \leq |\mathbb{N}|$, that is, if there exists an injective function $A \to \mathbb{N}$, or equivalently, if there exists a surjective function $\mathbb{N} \to A$. We say that A is *countably infinite* if it is both countable and infinite. A set is called *uncountable* if it is not countable.

Proposition 7.1. *Every infinite set has a countable subset.*

Proof. Let A be an infinite set. Suppose, by way of contradiction, that there does not exist an injective function from $\mathbb{N}$ to A.

For every $f : \mathbb{N} \to A$, define $F_f = \{n \in \mathbb{N} \mid f(n) = f(m) \text{ for some } m < n\}$. Since f is not injective, F_f is nonempty. Set $n_f = \min(F_f)$.

Define the set $M \subset \mathbb{N}$ by

$$M = \{n \in \mathbb{N} \mid \exists f : \mathbb{N} \to A \text{ such that } n = n_f.\}$$

Let $m = \max(M)$; then there exists $g : \mathbb{N} \to A$ such that $m = n_g$. Thus the function $g \restriction_{\mathbb{N}_m} : \mathbb{N}_m \to A$ is injective. Since A is infinite, $g \restriction_{\mathbb{N}_m}$ is not surjective, so there exists $a \in A$ such that $g(n) \neq a$ for every $n \in \mathbb{N}_m$. Define a function

$$h : \mathbb{N} \to A \quad \text{by} \quad h(n) = \begin{cases} g(n) & \text{if } n < m; \\ a & \text{if } n \geq m. \end{cases}$$

Now $n_h = m + 1$, so $m + 1 \in M$, contradicting $m = \max(M)$. □

The Hebrew *aleph* is written $\aleph$. Cantor define $\aleph_0$ to be the cardinality of the natural numbers: $\aleph_0 = |\mathbb{N}|$. As a corollary of the previous proposition, $\aleph_0$ is the smallest of the "transfinite" cardinals.

Corollary 7.2. *Let A be an infinite set. Then $|\mathbb{N}| \leq |A|$.*

Proof. Let B be a countable subset of A. Then the inclusion function

$$\text{inc} : B \to A \quad \text{given by} \quad \text{inc}(b) = b$$

is injective, so $|\mathbb{N}| = |B| \leq |A|$. □

Proposition 7.3. *Every subset of a countable set is countable.*

Proof. Let A be a countable set and let $B \subset A$. Since A is countable, there exists an injective function $f : A \to \mathbb{N}$. Then $f \restriction_B : B \to \mathbb{N}$ is also injective, so B is countable. □

Proposition 7.4. *Let A and B be countable sets. Then $A \cup B$ is countable.*

Proof. Since A and B are countable, there exist surjective functions $g : \mathbb{N} \to A$ and $h : \mathbb{N} \to B$. Define a function

$$f : \mathbb{N} \to A \cup B \quad \text{by} \quad f(n) = \begin{cases} g(\frac{n}{2}) & \text{if } n \text{ is even;} \\ h(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Then f is surjective, so $A \cup B$ is countable. □

Proposition 7.5. *Let A and B be countable sets. Then $A \times B$ is countable.*

Proof. Since A and B are countable, there exist injective functions $g : A \to \mathbb{N}$ and $h : B \to \mathbb{N}$. Define a function

$$f : A \times B \to \mathbb{N} \quad \text{by} \quad f(a, b) = 2^{g(a)} \cdot 3^{h(b)}.$$

To see that f is injective, suppose that $f(a_1, b_1) = f(a_2, b_2)$. Then $2^{g(a_1)}3^{h(b_1)} = 2^{g(a_2)}3^{h(b_2)}$. Thus $2^{g(a_1)-g(a_2)} = 3^{h(b_2)-h(b_1)}$, where without loss of generality $g(a_1) \geq g(a_2)$. If $g(a_1) > g(a_2)$, then 2 divides the left side and not the right; this is impossible, so $g(a_1) = g(a_2)$, and since g is injective, we must have $a_1 = a_2$. Similarly, $b_1 = b_2$. □

Proposition 7.6. *The set $\mathbb{Z}$ of integers is a countable set.*

Proof. Define a function

$$f : \mathbb{Z} \to \mathbb{N} \quad \text{by} \quad f(n) = \begin{cases} 1 & \text{if } n = 0; \\ 2n & \text{if } n > 0; \\ 2n + 1 & \text{if } n < 0. \end{cases}$$

Then f is injective, so $\mathbb{Z}$ is countable. □

Proposition 7.7. *The set $\mathbb{Q}$ of rational numbers is a countable set.*

Proof. Let $\mathbb{Z}^+$ denote the positive integers, $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. This is a subset of $\mathbb{Z}$, and is therefore countable. By Proposition 7.5, it suffices to find an injective function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$. Every rational number has a unique expression $\frac{p}{q}$ as a ratio of integers, where $\gcd(p, q) = 1$ and $q > 0$. This induces a function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}^+$ given by $\frac{p}{q} \mapsto (p, q)$. This function is bijective; therefore $\mathbb{Q}$ is countable. □

2. Cardinal Arithmetic

Let A and B be sets. We define the sum, product, and exponentiation of cardinal numbers to match that of finite numbers.

Define

$$|A| + |B| = |(A \times \{0\}) \cup (B \times \{1\})|.$$

Note that even if $A \cap B$ is nonempty, $A \times \{0\}$ and $B \times \{1\}$ are disjoint sets. So if A is any set with m elements and B is any set with n elements, then $(A \times \{0\}) \cup (B \times \{1\})$ is a set with $m + n$ elements.

Define

$$|A| \cdot |B| = |A \times B|.$$

Again, if A and B are finite with m and n elements respectively, then $A \times B$ has mn elements.

Define

$$|A|^{|B|} = |\mathcal{F}(B, A)|,$$

where $\mathcal{F}(B, A)$ denotes the set of all functions from B to A. This again agrees with the finite case.

We have seen that for any set X, there does not exist a surjective function from X to its power set $\mathcal{P}(X)$. Thus $|X| < |\mathcal{P}(X)|$. Actually, the next proposition shows that $|\mathcal{P}(X)| = 2^{|X|}$.

Proposition 7.8. *Let X be any set and let $T = \{0, 1\}$. Let $\mathcal{P}(X)$ denote the power set of X and let $\mathcal{F}(X, T)$ denote the set of all functions from X to T. Then $|\mathcal{P}(X)| = |\mathcal{F}(X, T)|$.*

Proof. Define a function

$$\Phi : \mathcal{F}(X, T) \to \mathcal{P}(X) \quad \text{by} \quad \Phi(f) = f^{-1}(1).$$

It suffices to show that Φ is bijective.

To see that Φ is injective, suppose that $\Phi(f_1) = \Phi(f_2)$, where $f_1 : X \to T$ and $f_2 : X \to T$. Then $f_1(x) = 1$ if and only if $f_2(x) = 1$. For $x \in X$, $f_i(x)$ is either 1 or 0, so if it is not 1, it is zero. Therefore $f_1(x) = 0$ if and only if $f_2(x) = 0$. So $f_1(x) = f_2(x)$ for every $x \in X$, that is, $f_1 = f_2$.

To see that Φ is surjective, let $A \in \mathcal{P}(X)$. Define a function

$$f : X \to T \quad \text{by} \quad f(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases}$$

Then $A = f^{-1}(1)$, so $\Phi(f) = A$. $\square$

3. Intervals

We define *intervals* of real numbers as follows:

(a) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$;
(b) $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$;
(c) $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$;
(d) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$;
(e) $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$;
(f) $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$;
(g) $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$;
(h) $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$;
(i) $(-\infty, \infty) = \mathbb{R}$.

Intervals of types **(a)**, **(e)**, **(g)**, and **(i)** are called *open intervals*, and intervals of types **(d)**, **(f)**, **(h**, and **(i)** are called *closed intervals*.

Proposition 7.9. *Any two intervals have the same cardinality.*

Proof. We show part of this and leave the remaining details to the reader.

First note that the function $x \mapsto \frac{x-a}{b-a}$ maps (a, b) bijectively onto $(0, 1)$. So all intervals of type **(a)** have the same cardinality.

Next consider the function $x \mapsto e^x$, which produces a bijective correspondence between $\mathbb{R}$ and $(0, \infty)$.

Finally consider the function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, which is also bijective. This demonstrates how all of the open intervals are equivalent. ☐

4. Base β Expansions

Let A be a set. A *sequence* in A is a function $a : \mathbb{Z}^+ \to A$. We write a_i to mean $a(i)$, and we write $\{a_i\}_{i=1}^{\infty}$, or simply $\{a_i\}$, to denote the function a.

Let β be an integer such that $\beta \geq 2$, and let $\mathbb{N}_\beta = \{0, 1, \ldots, \beta - 1\}$. Let $O = (0, 1)$ be the open unit interval in the real line. We are interested in relating the set of sequences in $\mathbb{N}_\beta$, which is denoted by $\mathcal{F}(\mathbb{Z}^+, \mathbb{N}_\beta)$, to the set O.

Define a function

$$\mu : \mathbb{Z} \to \mathbb{N}_\beta \quad \text{by} \quad \mu(n) = r,$$

where $n = \beta q + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < \beta$.

Define a function

$$\zeta : \mathbb{R} \to \mathbb{Z} \quad \text{by} \quad \zeta(x) = \max\{n \in \mathbb{N} \mid n \leq x\}.$$

For each $k \in \mathbb{Z}^+$, define a function

$$\delta_{\beta,k} : \mathbb{R} \to \mathbb{N}_\beta \quad \text{by} \quad \delta_{\beta,k}(x) = \mu(\zeta(\beta^k x)).$$

This induces a function

$$\delta_\beta : O \to \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_\beta) \quad \text{by} \quad \delta_\beta(x) = \{\delta_{\beta,k}(x)\}_{k=1}^{\infty}.$$

Then δ_β is an injective function, and we call $\delta_\beta(x)$ the *base β expansion* of x.

Construct a partial inverse to δ_β as follows.

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{N}_\beta$ and set $B = \{\sum_{i=1}^{k} \frac{a_i}{\beta^i} \mid k \in \mathbb{N}\}$. Then $B \subset O$, and in particular, B is a bounded set of real numbers. Set $b = \sup(B)$. For most sequences, $\delta_\beta(b) = \{a_i\}_{i=1}^{\infty}$.

Call a sequence $\{a_i\}_{i=1}^{\infty}$ in $\mathbb{N}_\beta$ a *duplicator* if there exists $N \in \mathbb{N}$ such that $a_i = \beta - 1$ for all $i > N$. These are the only sequences which are not in the image of the function δ_β. If $S = \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_\beta) \setminus \{$ duplicators $\}$, then $\delta_\beta : O \to S$ is bijective.

5. Uncountability

Proposition 7.10. *The set* $\mathbb{R}$ *of real numbers is an uncountable set.*

Proof. Since $O = (0,1) \subset \mathbb{R}$, it suffices to show that O is uncountable.

Let $\beta = 10$ so that we consider base 10 expansions of the elements in O, and let μ, ζ, and δ_β be as in the previous section.

Let $f : \mathbb{N} \to O$ be any function; we will show that f is not surjective. Set

$$a_i = \begin{cases} 3 & \text{if} \quad \beta_i(f(i)) \neq 3; \\ 6 & \text{if} \quad \beta_i(f(i)) = 3. \end{cases}$$

Set $B = \{\sum_{i=1}^{k} \frac{a_i}{10^i} \mid k \in \mathbb{N}\}$. Then $B \subset O$, and in particular, B is a bounded set of real numbers. Set $b = \sup(B)$. Then b is not in the image of f. $\square$

We can be even more precise than this.

Proposition 7.11. $|\mathbb{R}| = 2^{\aleph_0}$.

Proof. Again let $O = (0,1)$. Since $|\mathbb{R}| = |O|$, it suffices to prove that the cardinality of O equals that of $\mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2)$.

First construct a function

$$f : \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2) \to O \quad \text{by} \quad f(a_i) = \sup\left\{\sum_{i=1}^{k} \frac{a_i}{10^i} \mid k \in \mathbb{Z}^+\right\}.$$

This function is injective.

Next consider that $\delta_2 : O \to \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2)$ is injective.

By the Schroeder-Bernstein theorem, there exists a bijective function $O \to \mathcal{F}(\mathbb{Z}^+, \mathbb{N}_2)$. $\square$

CHAPTER 8

Lecture 8 - Sequences

1. Review

We described how the natural numbers can be build from axioms of set theory; how to construct the integers from the natural numbers, and how to construct the rationals from the integers.

We developed the real numbers as the set of cuts in the rational number line. This set supports addition, multiplication, and an ordering satisfying these properties:

(F1) $(a + b) + c = a + (b + c)$;
(F2) $a + 0 = a$;
(F3) $a + (-a) = 0$;
(F4) $a + b = b + a$;
(F5) $(ab)c = a(bc)$;
(F6) $a \cdot 1 = a$;
(F7) $a \cdot a^{-1} = 1$ for $a \neq 0$;
(F8) $ab = ba$;
(F9) $(a + b)c = ac + bc$;
(O1) $a \leq a$;
(O2) $a \leq b$ and $b \leq a$ implies $a = b$;
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$;
(O4) $a \leq b$ or $b \leq a$;
(O5) $a \leq b$ implies $a + c \leq b + c$;
(O6) $a \leq b$ implies $ac \leq bc$ for $c \geq 0$.
(CM) every set of real numbers bounded above has a least upper bound.

Property **(CM)** is equivalent to the lack of gaps in the real number line; this lack of gaps was proved using the Cantor-Dedekind Theorem. The Schroeder-Bernstein theorem helped show that there is a linear order on the cardinal numbers. It is the lack of gaps which insures that base β expansions produce real numbers, which leads to the proof the $|\mathbb{Q}| < |\mathbb{R}|$.

Exercise 8.1. Recommended practice exercises from the book:
Chapter 0 exercises 10,13,14,21,32,36,38,40

2. Triangle Inequality

Let $x \in \mathbb{R}$, and define the *absolute value* of x, denoted by $|x|$, by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Clearly $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$. We think of this as the distance between x and 0. Moreover, we think of $|x - a|$ as the distance between x and another real number a.

Proposition 8.1. *Let $a, b \in \mathbb{R}$. If $a \leq b$, then $-b \leq -a$.*

Proof. This uses property **(O5)**. Take $a \leq b$ and add $-b$ to both sides to get $a - b \leq 0$. Now add $-a$ to both sides to get $-b \leq -a$. □

Proposition 8.2 (Triangle Inequality)**.** *Let $a, b \in \mathbb{R}$. Then $|a + b| \leq |a| + |b|$.*

Proof. We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Repeated application of property **(O6)** yield

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Multiply both sides of the left inequality by -1 to obtain $-(a + b) \leq |a| + |b|$. Now $|a + b|$ is either $a + b$ or $-(a + b)$, and in either case, we see that $|a| + |b|$ is greater than it. □

3. Sequences

Let A be a set. A *sequence* in A is a function $a : \mathbb{Z}^+ \to A$. We write a_n to mean $a(n)$, and we write $\{a_n\}_{n=1}^{\infty}$, or simply $\{a_n\}$, to denote the function a. We will primarily be interested in sequences of real numbers, that is, sequences in $\mathbb{R}$.

4. Limit Points of Sequences

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $L \in \mathbb{R}$. We say that L is a *limit point* of $\{a_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}^+ \ni n \geq N \Rightarrow |a_n - L| < \epsilon.$$

In this case, we say that $\{a_n\}_{n=1}^{\infty}$ *converges* to L.

Proposition 8.3. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and let $L_1, L_2 \in \mathbb{R}$. If $\{a_n\}_{n=1}^{\infty}$ converges to L_1 and to L_2, then $L_1 = L_2$.*

Proof. Suppose not, and set $d = |L_1 - L_2|$; then d is positive. Let $\epsilon = \frac{d}{4}$. Then by definition of limit, there exist positive integers N_1 and N_2 such that $n \geq N_1$ implies that $|a_n - L_1| < \epsilon$, and $n \geq N_2$ implies that $|a_n - L_2| < \epsilon$.

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$,

$$\begin{aligned} d &= |L_1 - L_2| \\ &= |L_1 - a_n + a_n - L_2| \\ &= |L_1 - a_n| + |a_n - L_2| \qquad \text{by the Triangle Inequality} \\ &= |a_n - L_1| + |a_n - L_2| \\ &\leq \epsilon + \epsilon \\ &= \frac{d}{2}. \end{aligned}$$

This is a contradiction; thus $L_1 = L_2$. □

Thus limits are unique when they exist, justifying the article *the* limit instead of "a limit point". We write $L = \lim_{n\to\infty} a_n$ to say that $\{a_n\}_{n=1}^{\infty}$ converges to L.

If a sequence has a limit, we say that it is *convergent*; otherwise it is *divergent*.

Example 8.4. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. This gives $\frac{1}{N} < \epsilon$. Note that if $n \geq N$, then $1 \geq \frac{N}{n}$, and $\frac{1}{N} \geq \frac{1}{n}$. Thus for $n \geq N$ we have

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$. □

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The *image* of $\{a_n\}_{n=1}^{\infty}$ is the image of the sequence as a function, that is, it is the set

$$\{a_n \mid n \in \mathbb{Z}^+\}.$$

Note that there is much more information in a sequence than in its image; for example, the sequences $\{1 + (-1)^n\}_{n=1}^{\infty}$ and $\{0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, \dots\}$ have the same image; the common image is $\{0, 2\}$, a set containing two elements.

5. Bounded Sequences

We say that $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if there exists $a \in \mathbb{R}$ such that $a \geq a_n$ for every $n \in \mathbb{Z}^+$.

We say that $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if there exists $b \in \mathbb{R}$ such that $b \leq a_n$ for every $n \in \mathbb{Z}^+$.

We say that $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below. Equivalently, $\{a_n\}_{n=1}^{\infty}$ is bounded if there exists $M > 0$ such that $a_n \in [-M, M]$ for every $n \in \mathbb{Z}^+$.

Proposition 8.5. *Every convergent sequence is bounded.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L. Let N be so large that for $n \geq N$ we have $|a_n - L| < 1$. And $|L|$ to both sides of this inequality and apply the triangle inequality to get, for every $n \geq N$,

$$|a_n| \leq |a_n - L| + |L| < 1 + |L|.$$

There are only finitely many terms of the sequence between a_1 and a_{N-1}; set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\}.$$

Then $M \geq a_n$ for every $n \in \mathbb{Z}^+$, so $\{a_n\}_{n=1}^{\infty}$ is bounded. $\square$

CHAPTER 9

Lecture 9 - Arithmetic of Sequences

1. Definitions of Sup and Inf

Recall the following definitions.
Let $S \subset \mathbb{R}$ and let $x \in \mathbb{R}$.
We say that $x = \max S$ if

(a) $x \geq s$ for every $s \in S$;
(b) $x \in S$.

We say that $x = \min S$ if

(a) $x \leq s$ for every $s \in S$;
(b) $x \in S$.

We say that $x = \sup S$ if

(a) $x \geq s$ for every $s \in S$;
(b) $a \geq s$ for every $s \in S \Rightarrow a \geq s$.

We say that $x = \sup S$ if

(a) $x \geq s$ for every $s \in S$;
(b) $a \geq s$ for every $s \in S \Rightarrow a \geq s$.

2. Examples of Sup and Inf

Example 9.1. Let S be a nonempty bounded subsets of $\mathbb{R}$. Show that $\inf S \leq \sup S$. What can be said if $\inf S = \sup S$?

Proof. Since S is nonempty, there exists $s \in S$. Then $\inf S \leq s$ and $s \leq \sup S$. By transitivity of order, $\inf S \leq \sup S$.

If $\inf S = \sup S$, then S contains only one element. □

Example 9.2. Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show if $S \subset T$, the $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Proof. Let $s \in S$. Then $s \in T$, so $\inf T \leq s$. Thus $\inf T$ is a lower bound for S, so $\inf T \leq \inf S$. Similarly, $\sup S \leq \sup T$. That $\inf S \leq \sup S$ is true is above. □

Example 9.3. Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Proof. Either $\max\{\sup S, \sup T\} = \sup S$ or $\max\{\sup S, \sup T\} = \sup T$.

Suppose that $\max\{\sup S, \sup T\} = \sup S$; in this case, $\sup T \leq \sup S$. Since $S \subset S \cup T$, we have $\sup S \leq \sup(S \cup T)$ by part (a).

Now let $x \in S \cup T$. Then x is either in S or T. If $x \in S$, then $x \leq \sup S$. If $x \in T$, then $x \leq \sup T \leq \sup S$. Thus $\sup S$ is an upper bound for $S \cup T$. Therefore $\sup(S \cup T) \leq \sup S$.

Since $\sup S \leq \sup(S \cup T)$ and $\sup(S \cup T) \leq \sup S$, it follows that $\sup S = \sup(S \cup T)$.

Finally, if $\max\{\sup S, \sup T\} = \sup T$, the above proof is valid, with the roles of S and T reversed. □

Example 9.4. Show that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Proof. Let $b = \max\{a, \frac{1}{a}\}$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > b$. Since $a \leq b$, we have $a < n$. Also since $\frac{1}{a} \leq b$, we have $\frac{1}{a} < n$. Thus by Theorem 3.2.(vii), we have $\frac{1}{n} < a$. □

Example 9.5. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that there exist infinitely many rational numbers between a and b.

Proof. Suppose not. The the set $S = (a, b) \cap \mathbb{Q}$ is finite, so it has a minimum, say $c = \min S$. But then Theorem 4.7 tells us that there exists $d \in \mathbb{Q}$ such that $a < d < c$. But then $d < b$, so $d \in S$. This contradicts that $c = \min S$. □

Example 9.6. Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let

$$S = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup S = \sup A + \sup B$.
(b) Show that $\inf S = \inf A + \inf B$.

Lemma 9.7. *Let $A \subset \mathbb{R}$ be bounded above and suppose that $x < \sup A$. Then there exists $a \in A$ such that $x < a$.*

Proof of Lemma. Suppose not; then for every $a \in A$, we have $a \leq x$. Then x is an upper bound for A, so $\sup A \leq x$, contrary to our assumption on x. □

Proof of Example. We prove **(a)**; the proof for **(b)** is symmetric. It suffices to show that $\sup S \leq \sup A + \sup B$ and that $\sup A + \sup B \leq \sup S$.

Let $s \in S$. Then $s = a + b$ for some $a \in A$ and $b \in B$. Then $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound for S, so $\sup S \leq \sup A + \sup B$.

Suppose that $\sup S < \sup A + \sup B$. Then $\sup S - \sup B < \sup A$, so there exists $a \in A$ such that $\sup S - \sup B < a$. From this, $\sup S - a < \sup B$, so there exists $b \in B$ such that $\sup S - a < b$. Let $s = a + b \in S$. We have $\sup S < s$, a contradiction. Therefore $\sup A + \sup B \leq \sup S$. □

3. Arithmetic of Sequences

Lemma 9.8. *Let* $a, b \in \mathbb{R}$. *Then* $|ab| = |a||b|$.

Reason. Break this into four cases and see the result. □

Proposition 9.9. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a convergent sequence of real numbers, and let* $k \in \mathbb{R}$. *Then*

$$k \lim_{n\to\infty} s_n = \lim_{n\to\infty} (ks_n).$$

Proof. Let $\epsilon > 0$, and set $s = \lim_{n\to\infty} s_n$. Since $s_n \to s$, there exists $N \in \mathbb{Z}^+$ such that

$$|s_n - s| < \frac{\epsilon}{k}.$$

Then

$$|ks_n - ks| < \epsilon.$$

□

Proposition 9.10. *Let* $\{s_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *be convergent sequences of real numbers. Then the sequence* $\{s_n + t_n\}_{n=1}^{\infty}$ *converges, and*

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$

Proposition 9.11. *Let* $\{s_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *be convergent sequences of real numbers. Then the sequence* $\{s_n t_n\}_{n=1}^{\infty}$ *converges, and*

$$\lim_{n\to\infty} (s_n t_n) = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n).$$

Proposition 9.12. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a convergent sequence of nonzero real numbers. Then*

$$\frac{1}{\lim_{n\to\infty} s_n} = \lim_{n\to\infty} (\frac{1}{s_n}).$$

Lemma 9.13. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence of nonzero real numbers such that* $\lim_{n\to\infty} |s_n|$ *converges to a positive real number. Then there exists* $m > 0$ *such that* $|s_n| > m$ *for all* n.

CHAPTER 10

Lecture 10 - Monotone Sequences

1. Infinity

The extended real numbers are $\mathbb{R} \cup \{\pm\infty\}$.

If A is unbounded above, then $\sup A = \infty$.

If A is unbounded below, then $\inf A = -\infty$.

If $\lim a_n = \pm\infty$, we say that "diverges to +- infinity".

Arithmetic of infinity based on sequences can be developed.

We say that $\lim s_n > 0$ if $\{s_n\}_{n=1}^{\infty}$ converges to a positive real number, or if $\{s_n\}_{n=1}^{\infty}$ diverges to ∞.

Proposition 10.1. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim s_n > 0$. Then there exists $N \in \mathbb{Z}^+$ and $P > 0$ such that if $n \geq N$, then $s_n > P$.*

Proof. If $s_n \to +\infty$, this follows directly from the definition. Thus assume that $\lim s_n = L > 0$, and set $\epsilon = \frac{L}{2}$. Let N be so large that $|s_n - L| < \epsilon$ for $n \geq N$. Then for such n, $s_n > L - \epsilon$. Let $P = L - \epsilon$. □

Proposition 10.2. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be sequences of positive real numbers such that $\lim s_n = +\infty$ and $\lim t_n > 0$. Then*

- **(a)** $\lim(s_n + t_n) = +\infty$;
- **(b)** $\lim(s_n t_n) = +\infty$.

Proof. Let $M > 0$.

Since $\lim t_n > 0$, there exists $N_1 \in \mathbb{Z}^+$ and $P > 0$ such that if $n \geq N_1$ then $t_n > P$.

Since $\lim s_n = +\infty$, there exists $N_2 \in \mathbb{Z}^+$ such that if $n \geq N_2$ then $s_n > \frac{M}{P}$.

Set $N = \max N_1, N_2$; for $n \geq N$, we have

$$s_n t_n > \frac{M}{P} P = M.$$

□

Proposition 10.3. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then*

$$\lim s_n = +\infty \Leftrightarrow \lim \frac{1}{s_n} = 0.$$

Proof. To show an if and only if statement, we show both directions.

($\Rightarrow$) Suppose that $\lim s_n = +\infty$. Let $\epsilon > 0$, set $M = \frac{1}{\epsilon}$. Since $s_n \to +\infty$, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $s_n > M$. Then for $n \geq N$, we have $|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \epsilon$.

($\Leftarrow$) Suppose that $\lim \frac{1}{s_n} = 0$. Let $M > 0$ and set $\epsilon = \frac{1}{M}$. Since $\frac{1}{s_n} \to 0$, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|\frac{1}{s_n} - 0| < \epsilon$. Since s_n is positive, this is the same as $\frac{1}{s_n} < \epsilon$, which implies that $s_n > M$. Thus $s_n \to +\infty$. $\square$

2. Monotone Sequence

Let $\{s_n\}_{n=1}^\infty$ be a sequence of real numbers.

We say that $\{s_n\}_{n=1}^\infty$ is *increasing* if

$$m \leq n \Rightarrow s_m \leq s_n.$$

We say that $\{s_n\}_{n=1}^\infty$ is *decreasing* if

$$m \leq n \Rightarrow s_m \geq s_n.$$

We say that $\{s_n\}_{n=1}^\infty$ is *monotone* if it is either increasing or decreasing.

Proposition 10.4. *Let $\{s_n\}_{n=1}^\infty$ be a monotone sequence.*

(a) *If $\{s_n\}_{n=1}^\infty$ is bounded, then it converges.*
(b) *If $\{s_n\}_{n=1}^\infty$ is unbounded and increasing, then it diverges to $+\infty$.*
(c) *If $\{s_n\}_{n=1}^\infty$ is unbounded and decreasing, then it diverges to $-\infty$.*

Proof.

(a) Suppose that $\{s_n\}_{n=1}^\infty$ is bounded. Also assume that it is increasing; the proof for decreasing will be analogous. Let $S = \{s_n \mid n \in \mathbb{Z}^+\}$ be the image of the sequence, and set $u = \sup S$. Since S is bounded, $u \in \mathbb{R}$. Clearly $s_n \leq u$ for every $n \in \mathbb{Z}^+$. We show that $\lim s_n = u$.

Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists $s \in S$ such that $u - \epsilon < s < u$. Now $s = s_N$ for some $N \in \mathbb{Z}^+$, and since $\{s_n\}_{n=1}^\infty$ is increasing, we have $u - \epsilon < s_n < u$ for every $n \geq N$. Thus $|s_n - u| < \epsilon$ for $n \geq N$; this shows that $s_n \to u$.

(b) Let $M > 0$. ☐

Proposition 10.5. *Let $\{s_n\}_{n=1}^\infty$ be a sequence of real numbers. Set*

$$u_N = \sup\{s_n \mid n \geq N\}$$

and

$$v_N = \inf\{s_n \mid n \geq N\}.$$

Then $\{u_n\}_{n=1}^\infty$ is a decreasing sequence and $\{v_n\}_{n=1}^\infty$ is an increasing sequence.

Proof. ☐

Let $\{s_n\}_{n=1}^\infty$ be a sequence of real numbers. Define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n \geq N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n \geq N\}.$$

Proposition 10.6. *Let $\{s_n\}_{n=1}^\infty$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^\infty$ converges if and only if $\liminf s_n = \limsup s_n$, in which case $\liminf s_n = \lim s_n = \limsup s_n$.*

3. Cluster Points of Sequences

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and let $c \in \mathbb{R}$. We say that c is a *cluster point* of $\{s_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \, \forall N \in \mathbb{Z}^+ \, \exists n \geq N \, \ni \, |s_n - c| < \epsilon.$$

Proposition 10.7. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let C be the set of cluster points of $\{s_n\}_{n=1}^{\infty}$. Then*

- **(a)** $\limsup s_n \in C$;
- **(b)** $\liminf s_n \in C$;
- **(c)** $\sup C = \limsup s_n$;
- **(d)** $\inf C = \liminf s_n$.

Proof. Since $\{s_n\}_{n=1}^{\infty}$ is bounded, $\limsup s_n$ and $\liminf s_n$ exist as real numbers. Let $s = \limsup s_n$; we will prove **(a)** and **(c)**, the proofs for **(b)** and **(d)** being analogous.

For **(a)**, suppose not; then $s \notin C$. That is, there exists $\epsilon > 0$ and $N \in \mathbb{Z}^+$ such that $|s_n - s| > \epsilon$ for all $n \geq N$. Now either there exists $n \geq N$ such that $s_n > s + \epsilon$, or for every $n \geq N$, $s_n < s - \epsilon$.

In the first case, s cannot be an upper bound for S, a contradiction. In the second case, $s - \epsilon$

□

CHAPTER 11

Lecture 11 - Lim Sup and Lim Inf

1. lim sup and lim inf

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n \geq N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n \geq N\}.$$

The sequence $\{\sup\{s_n \mid n \geq N\}_{N=1}^{\infty}$ is decreasing and the sequence $\{\inf\{s_n \mid n \geq N\}_{N=1}^{\infty}$ is increasing, so they both converge, or diverge to $\pm\infty$.

Proposition 11.1. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then $\liminf s_n \leq \limsup s_n$.*

Lemma 11.2. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be sequences such that $s_n \leq t_n$ for every $n \in \mathbb{N}$. If they both converge, we have $\lim s_n \leq \lim t_n$.*

proof of Lemma. Let $s = \lim s_n$ and $t = \lim t_n$; suppose by way of contradiction that $t < s$. Set $\epsilon = \frac{t-s}{2}$; then there exists $N_1 \in \mathbb{Z}^+$ such that $n \geq N_1$ implies $|s_n - s| < \epsilon/2$, and there exists $N_2 \in \mathbb{Z}^+$ such that $n \geq N_2$ implies $|t_n - s| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then by and application of the triangle inequality, $t_n < s_n$, a contradiction. □

proof of Proposition. For every $N \in \mathbb{Z}^+$, we have $\inf\{s_n \mid n \geq N\} \leq \sup\{s_n \mid n \geq N\}$. Thus the result is immediate from the lemma. □

Proposition 11.3. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. Then* $\{s_n\}_{n=1}^{\infty}$ *converges if and only if* $\liminf s_n = \limsup s_n$*, in which case* $\liminf s_n = \lim s_n = \limsup s_n$*.*

Lemma 11.4. *Let* $x, y \in \mathbb{R}$*. If* $x \leq y + \epsilon$ *for every* $\epsilon > 0$*, then* $x \leq y$*.*

Proof. Suppose that $x > y$, and let $\epsilon = \frac{x-y}{2}$. Then $y + \epsilon = x - \epsilon$, so $x > y + \epsilon$. □

Proof.

(⇒) Suppose that $\{s_n\}_{n=1}^{\infty}$ converges to a real number s. Let $\epsilon > 0$. We wish to show that $\limsup s_n \leq s + \epsilon$ for every $\epsilon > 0$, whence $\limsup s_n \leq s$.

Since $s_n \to s$, there exists $N \in \mathbb{Z}^+$ such that $|s_n - s| < \epsilon$ for $n \geq N$. Then $\sup\{s_n \mid n \geq N\} < s + \epsilon$. Since $\{\sup\{s_n \mid n \geq N\}\}_{N=1}^{\infty}$ is a decreasing sequence, we have $\limsup s_n < s + \epsilon$. Therefore $\limsup s_n \leq s$.

Similarly, $s \leq \liminf s_n$, so

$$s \leq \liminf s_n \leq \limsup s_n \leq s,$$

so

$$\liminf s_n = s = \limsup s_n.$$

(⇐) Now suppose that $\liminf s_n = \limsup s_n$, and label this common value s. We want to show that $\lim s_n = s$.

Let $\epsilon > 0$. Since $s = \limsup s_n$, there exists $N_1 \in \mathbb{Z}^+$ such that

$$|\sup\{s_n \mid n \geq N_1\} - s| < \epsilon.$$

In particular, $\sup\{s_n \mid n \geq N_1\} < s + \epsilon$, so $s_n < s + \epsilon$ for $n \geq N_1$. Similarly, since $s = \liminf s_n$, there exists $N_2 \in \mathbb{Z}^+$ such that $s_n > s - \epsilon$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have $s - \epsilon < s_n < s + \epsilon$, that is, $|s_n - s| < \epsilon$. Thus $s_n \to s$. □

CHAPTER 12

Lecture 12 - Cauchy Sequences

1. Cauchy Sequences

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that $\{s_n\}_{n=1}^{\infty}$ is a *Cauchy sequence* if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}^+ \; \ni \; m, n \geq N \Rightarrow |s_m - s_n| < \epsilon.$$

Proposition 12.1. *Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{s_n\}_{n=1}^{\infty}$ is bounded.*

Proof. Since $\{s_n\}_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|s_m - s_n| < 1$. In particular, for every $n \geq N$, we have $|s_n - s_N| < 1$. Set

$$M = \max\{s_1, s_2, \dots, s_{N-1}, s_N + 1\}.$$

Then $s_n \in [-M, M]$ for every $n \in \mathbb{Z}^+$. □

Proposition 12.2. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^{\infty}$ is convergent if and only if it is a Cauchy sequence.*

Proof. We prove each direction of the double implication.

($\Rightarrow$) Assume that the sequence is convergent. Let $\epsilon > 0$, and set $s = \lim s_n$. Then there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|s_n - s| < \epsilon/2$. Then for $m, n \geq N$, we have

$$\begin{aligned} |s_m - s_n| &= |s_m - s + s - s_n| \\ &= |s_m - s| + |s_n - s| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

($\Leftarrow$) Assume that the sequence is a Cauchy sequence. Then it is bounded, and so its limit superior and inferior exist as real numbers. By a previous proposition, it suffices to show that $\liminf s_n = limsups_n$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|s_m - s_n| < \epsilon$. In particular, $|s_n - s_N| < \frac{\epsilon}{2}$ for all $n \geq N$, so $s_N + \frac{\epsilon}{2}$ is an upper bound for $\{s_n \mid n \geq N\}$. Thus $\sup\{s_n \mid n \geq N\} \leq s_N + \frac{\epsilon}{2}$, and therefore $\limsup s_n \leq s_N + \frac{\epsilon}{2}$. Similarly $\liminf s_n \geq s_N - \frac{\epsilon}{2}$. Rearranging these inequalities gives

$$\limsup s_n - \frac{\epsilon}{2} \leq s_N \leq \liminf s_n + \frac{\epsilon}{2},$$

or

$$\limsup s_n - \liminf s_m < \epsilon.$$

Since ϵ is arbitrary, we have $\limsup s_n = \liminf s_n$. □

2. Subsequences

Let $s : \mathbb{Z}^+ \to \mathbb{R}$ be a sequence of real numbers. A *subsequence* of s is the composition $s \circ n$ of s with a strictly increasing sequence $n : \mathbb{Z}^+ \to \mathbb{Z}^+$ of positive integers.

If we denote the sequence s by $\{s_n\}_{n=1}^\infty$ and the sequence n by $\{n_k\}_{k=1}^\infty$, then we denote the subsequence by $\{s_{n_k}\}_{k=1}^\infty$.

Note that since the function $n : \mathbb{Z}^+ \to \mathbb{Z}^+$ is strictly increasing, it is injective. Thus if $N \in \mathbb{Z}^+$, there exists $k \in \mathbb{Z}^+$ such that $n_k \geq N$; otherwise, we would have an injective function from an infinite set into the finite set $\{m \in \mathbb{Z}^+ \mid m < N\}$.

Proposition 12.3. *Let $\{s_n\}_{n=1}^\infty$ be a sequence of real numbers and let $s \in \mathbb{R}$. Then $\{s_n\}_{n=1}^\infty$ converges to s if and only if every subsequence of $\{s_n\}_{n=1}^\infty$ converges to s.*

Proof. We prove both directions.

($\Leftarrow$) Note that a sequence is a subsequence of itself. Thus if every subsequence of $\{s_n\}_{n=1}^\infty$ converges to s, then in particular the sequence itself converges to s.

($\Rightarrow$) Suppose that $\lim s_n = s$. Let $\{s_{n_k}\}$ be a subsequence of $\{s_n\}_{n=1}^\infty$, and let $\epsilon > 0$. Then there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|s_n - s| < \epsilon$. We know that there exists $K \in \mathbb{Z}^+$ such that $n_K \geq N$; moreover, since $\{n_k\}$ is strictly increasing, if $k \geq K$, then $n_k \geq n_K \geq N$. Therefore, for $k \geq K$, we have $|s_{n_k} - s| < \epsilon$. □

Proposition 12.4. *Let $\{s_n\}_{n=1}^\infty$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^\infty$ has a monotonic subsequence.*

Proof. Say that the i^{th} term of $\{s_n\}_{n=1}^\infty$ is *dominant* if $s_j < s_i$ for every $j > i$.

Case 1: There are infinitely many dominant terms. In this case, set

$$n_1 = \min\{n \in \mathbb{Z}^+ \mid s_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{Z}^+ \mid s_n \text{ is dominant and } n > n_k\};$$

this set is nonempty by the hypothesis of this case. Then $\{s_{n_k}\}$ is a decreasing sequence.

Case 2: There are finitely many dominant terms. In this case, set

$$n_0 = max\{n \in \mathbb{Z}^+ \mid s_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{Z}^+ \mid s_n > s_{n_k} \text{ and } n > n_k\};$$

this set is nonempty because s_{n_0} was the last dominant term. Now $\{s_{n_k}\}$ is an increasing sequence. □

Corollary 12.5. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. It is clear that if a sequence is bounded, then every subsequence is also bounded. Thus a bounded sequence has a bounded monotonic subsequence, which must converge. □

3. Cluster Points and Subsequential Limits

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$.

We say that c is a *cluster point* of $\{s_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{Z}^+ \; \exists n \geq N \;\ni\; |s_n - c| < \epsilon.$$

We say that c is a *subsequential limit* of $\{s_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} s_{n_k} = c$.

Proposition 12.6. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$. Then c is a cluster point if and only if c is a subsequential limit.*

Proof. Exercise. □

4. Neighborhoods

Let $x_0 \in \mathbb{R}$. An *ϵ-neighborhood* of x_0 is an open interval of the form $(x_0 - \epsilon, x_0 + \epsilon)$, where $\epsilon > 0$.

More generally, a *neighborhood* of x_0 is a subset $Q \subset \mathbb{R}$ such that there exists $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset Q$.

A *deleted neighborhood* of x_0 is a set of the form $Q \setminus \{x_0\}$, where Q is a neighborhood of x_0.

Proposition 12.7. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $s \in \mathbb{R}$. Then s is the limit of $\{s_n\}_{n=1}^{\infty}$ if and only if every neighborhood of s contains s_n for all but finitely many n.*

Proof. Gaughan page 35 Lemma. □

Proposition 12.8. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$. Then c is a cluster point of $\{s_n\}_{n=1}^{\infty}$ if and only if every neighborhood of c contains s_n for infinitely many n.*

Proof. Exercise. □

CHAPTER 13

Lecture 13 - Open and Closed Sets

1. Neighborhoods

Let $x_0 \in \mathbb{R}$. An ϵ-*neighborhood* of x_0 is an open interval of the form $(x_0 - \epsilon, x_0 + \epsilon)$, where $\epsilon > 0$.

More generally, a *neighborhood* of x_0 is a subset $Q \subset \mathbb{R}$ such that there exists $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset Q$.

A subset $U \subset \mathbb{R}$ is called *open* if

$$\forall u \in U \exists \epsilon > 0 \ni |x - u| < \epsilon \Rightarrow x \in U.$$

Or, in other words, U is open if every point in U is surrounded by an ϵ-neighborhood which is completely contained in U.

Proposition 13.1. *Let $\mathcal{T}$ denote the collection of all open subsets of $\mathbb{R}$. Then*

(a) $\varnothing \in \mathcal{T}$ *and* $\mathbb{R} \in \mathcal{T}$;
(b) *if* $\mathcal{O} \subset \mathcal{T}$, *then* $\cup\mathcal{O} \in \mathcal{T}$;
(c) *if* $\mathcal{O} \subset \mathcal{T}$ *is finite, then* $\cap\mathcal{O} \in \mathcal{T}$.

Proof.

(a) The condition for openness is vacuously satisfied by the empty set. For $\mathbb{R}$, consider $x \in \mathbb{R}$. Then $(x - 1, x + 1) \subset \mathbb{R}$. Thus $\mathbb{R}$ is open.

(b) Let $\mathcal{O} \subset \mathcal{T}$; that is, $\mathcal{O}$ is a collection of open sets. Select $x \in \cup\mathcal{O}$. Then $x \in U$ for some $U \in \mathcal{O}$. Since U is open, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Since $U \subset \cup\mathcal{O}$, it follows that $(x - \epsilon, x + \epsilon) \subset \cup\mathcal{O}$. Thus $\cup\mathcal{O}$ is open.

(c) Let $\mathcal{O} \subset \mathcal{T}$ be a finite collection of open sets. Since $\mathcal{O}$ is finite, we may write $\mathcal{O} = \{U_1, U_2, \ldots, U_n\}$, where U_i is an open set for $i = 1, \ldots, n$. If $\cap\mathcal{O}$ is empty, we are done, so assume that it nonempty, and select $x \in \cap\mathcal{O}$. For each i, there exists ϵ_i such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Set $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $(x - \epsilon, x + \epsilon) \subset \cap\mathcal{O}$. Thus $\cap\mathcal{O}$ is open. □

Lemma 13.2. *Let $\mathcal{O}$ be a collection of open intervals. If $\cap\mathcal{O}$ is nonempty, then $\cup\mathcal{O}$ is an open interval.*

Proof. By hypothesis, there exists $x \in \cap\mathcal{O}$. Write $\mathcal{O}$ as a family of sets:

$$\mathcal{O} = \{O_\alpha \mid \alpha \in A\},$$

where A is an indexing set. Now O_α is an open interval; we label its endpoints by letting $O_\alpha = (a_\alpha, b_\alpha)$, where $a_\alpha, b_\alpha \in \overline{\mathbb{R}}$. Set

$$a = \inf\{a_\alpha \mid \alpha \in A\} \quad \text{and} \quad b = \sup\{b_\alpha \mid \alpha \in A\}.$$

Claim: $\cup\mathcal{O} = (a, b)$. We prove both directions of containment.

($\subset$) Let $y \in \cup\mathcal{O}$. Then $y \in O_\alpha$ for some α. Thus $a \leq a_\alpha < y < b_\alpha \leq b$, so $y \in (a, b)$.

($\supset$) Let $y \in (a, b)$. Assume that $y \leq x$; the proof for $y \geq x$ is analogous. Now $a < y$, and since $a = \inf\{a_\alpha \mid \alpha \in A\}$, so there exists $\alpha \in A$ such that $a \leq a_\alpha < y$. Also $x \in O_\alpha$ so $a_\alpha < y \leq x < b_\alpha$; thus $y \in (a_\alpha, b_\alpha) = O_\alpha$, and $y \in \cup\mathcal{O}$. □

Proposition 13.3. *Let $U \subset \mathbb{R}$. Then U is open if and only if there exists a countable collection $\mathcal{O}$ of disjoint open intervals such that $U = \cup\mathcal{O}$.*

Proof. Put a relation on U by defining $u_1 \sim u_2$ if there exists an open interval O such that $u_1, u_2 \in O$ and $O \subset U$.

Claim 1: This is an equivalence relation. We wish to show that $\sim$ is reflexive, symmetric, and transitive.

Reflexive Let $u \in U$. Since U is open, there exists $\epsilon > 0$ such that $(u-\epsilon, u+\epsilon) \subset U$. Let $O = (u - \epsilon, u + \epsilon)$; then $u \in O$ and $O \subset U$, so $u \sim u$.

Symmetric Let $u_1, u_2 \in U$, and assume $u_1 \sim u_2$. Then there exists an open interval O such that $u_1, u_2 \in O$ and $O \subset U$. But then $u_2, u_1 \in O$, so $u_2 \sim u_1$.

Transitive Let $u_1, u_2, u_3 \in U$, and assume that $u_1 \sim u_2$ and $u_2 \sim u_3$. Then there exist open intervals $O_1, O_2 \subset U$ such that $u_1, u_2 \in O_1$ and $u_2, u_2 \in O_2$. Now $u_2 \in O_1 \cap O_2$, so by the Lemma, $O_1 \cup O_2$ is an interval contained in U and containing u_1 and u_3. Thus $u_1 \sim u_3$.

Claim 2: The equivalence classes of this equivalence relation are open intervals. Let $u \in U$ and let $\overline{u}$ denote the equivalence class of u. For every $v \in \overline{U}$ there exists and open interval O_v such that $u, v \in O_v$ and $O_v \subset U$. Let $\mathcal{O} = \{O_v \mid v \in \overline{u}\}$. Then $u \in \cap\mathcal{O}$, so $\cup\mathcal{O}$ is an open interval; it suffices to show that $\overline{u} = \cup\mathcal{O}$. Clearly $\overline{u} \subset \cup\mathcal{O}$. Moreover, if $x \in \cup\mathcal{O}$, then $x \in O_v$ for some v, so $x \sim u$ and $x \in \overline{u}$. Thus $\overline{u} = \cup\mathcal{O}$.

Claim 3: Distinct equivalence classes have empty intersection. This is true for every equivalence relation.

Claim 4: There are only countably many equivalence classes. Let $\mathcal{O} = \{O_\alpha \mid \alpha \in A\}$ be the collection of equivalence classes, where A is some indexing set. Let $O_\alpha = (a_\alpha, b_\alpha)$. We have seen that there exists $q_\alpha \in \mathbb{Q}$ such that such that $a_\alpha < q_\alpha < b_\alpha$. Let $Q = \{q_\alpha \mid \alpha \in A\}$. Then $|\mathcal{O}| = |A| = |Q| \leq |\mathbb{Q}|$; since $\mathbb{Q}$ is countable, so is $\mathcal{O}$. □

2. Closed Sets

A subset $F \subset \mathbb{R}$ is *closed* if its complement $\mathbb{R} \smallsetminus F$ is open.

Proposition 13.4. *Let $\mathcal{F}$ denote the collection of all closed subsets of $\mathbb{R}$.*

- **(a)** $\varnothing \in \mathcal{F}$ *and* $\mathbb{R} \in \mathcal{F}$*;*
- **(b)** *if* $\mathcal{C} \subset \mathcal{F}$*, then* $\cap\mathcal{C} \in \mathcal{F}$*;*
- **(c)** *if* $\mathcal{C} \subset \mathcal{F}$ *is finite, then* $\cup\mathcal{C} \in \mathcal{T}$.

Proof. Apply DeMorgan's Laws to Proposition 13.1. □

Proposition 13.5. *Let $F \subset \mathbb{R}$. Then F is closed if and only if every sequence in F which converges in $\mathbb{R}$ has a limit in F.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that F is closed, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence in F which converges to $a \in \mathbb{R}$. We wish to show that $a \in F$. Suppose not; then $a \in \mathbb{R} \smallsetminus F$. This set is open, so there exists $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset \mathbb{R} \smallsetminus F$. Thus there exists $N \in \mathbb{Z}^+$ such that $a_n \in \mathbb{R} \smallsetminus F$ for all $n \geq N$. This contradicts that the sequence is in F.

($\Leftarrow$) Suppose that F is not closed; we wish to construct a sequence in F which converges to a point not in F. Since F is not closed, then $\mathbb{R} \smallsetminus F$ is not open. This means that there exists a point $x \in \mathbb{R} \smallsetminus F$ such that for every $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ is not a subset of $\mathbb{R} \smallsetminus F$; that is, $(x - \epsilon, x + \epsilon)$ contains a point in F. For $n \in \mathbb{Z}^+$, let $x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap F$. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence in F, but $\lim_{n \to \infty} x_n = x \notin F$. □

3. Accumulation Points

A *deleted neighborhood* of x_0 is a set of the form $Q \setminus \{x_0\}$, where Q is a neighborhood of x_0.

Let $S \subset \mathbb{R}$. An *accumulation point* of S is a point $s \in \mathbb{R}$ such that every deleted neighborhood of s contains an element of S.

We note that an accumulation point of a set S may or may not be an element of S.

Proposition 13.6. *Let $F \subset \mathbb{R}$. Then F is closed if and only if F contains all of its accumulation points.*

Proof. Prove both directions.

($\Rightarrow$) Suppose F is closed, and let $x \in \mathbb{R}$. Suppose $x \notin F$; we show that x is not an accumulation point of F. Since $x \in F$, then $x \in \mathbb{R} \setminus F$, which is open. Therefore there exists $\epsilon > 0$ such that $U = (x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus F$. Then $U \setminus \{x\}$ is a deleted neighborhood of x whose intersection with F is empty, and x is not an accumulation point of F.

($\Leftarrow$) Suppose F contains all of its accumulation points. We show that the complement of F is open. Let $x \in \mathbb{R} \setminus F$. Then x is not an accumulation point of F. Then there exists a deleted neighborhood U of x such that $U \subset \mathbb{R} \setminus F$. This neighborhood contains a deleted epsilon neighborhood, say $(x-\epsilon, x+\epsilon) \setminus \{x\}$. This set is in the complement of F, and since $x \notin F$, we have $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus F$. Thus $\mathbb{R} \setminus F$ is open, so F is closed. □

Theorem 13.7 (Bolzano-Weierstrass Theorem)**.** *Every bounded infinite set of real numbers has an accumulation point.*

Proof. Let S be a bounded infinite set. Since S is infinite, there exists an injective function $s : \mathbb{Z}^+ \to S$; view this as a sequence $\{s_n\}_{n=1}^{\infty}$. This sequence has a monotonic subsequence, say $\{s_{n_k}\}$, which is also bounded and hence convergent, say to $s \in \mathbb{R}$. Suppose that $s = s_{n_K}$ for some K; then, since s is the limit and the sequence is monotonic, it is easy to see that $s_{n_k} = s$ for every $k \geq K$. This contradicts that the sequence was injective. Thus $s \neq s_{n_k}$ for every $k \in \mathbb{Z}^+$.

Now for every $\epsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that $|s_{n_K} - s| < \epsilon$; that is, $s_{n_K} \in (s - \epsilon, s + \epsilon)$, and $s_{n_K} \neq s$. Thus s is an accumulation point for S. □

CHAPTER 14

Lecture 14

It was a nice day outside.

CHAPTER 15

Lecture 15 - Limits of Functions

1. Limit of a Function

Let $D \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and let $L \in \mathbb{R}$. We say that L is the *limit* of f at x_0, and write $L = \lim_{x \to x_0} f(x)$, if

$$\forall \epsilon > 0 \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Proposition 15.1. *Let $D \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and let $L \in \mathbb{R}$. Then $\lim_{x \to x_0} f(x) = L$ if and only if every deleted neighborhood of x_0 is mapped by f into a neighborhood of L.*

2. Main Examples

Do you know your asymptote from a hole in the graph?

Let $D = (-\infty, 0) \cup (0, \infty)$ and consider these examples of $f : D \to \mathbb{R}$.

Example 15.2 (Asymptote)**.** Let $f(x) = \frac{1}{x}$.

Example 15.3 (Hole in the Graph)**.** Let $f(x) = \frac{x^2}{x}$.

Example 15.4 (Jump)**.** Let $f(x) = \frac{|x|}{x}$.

Example 15.5 (Oscillation)**.** Let $f(x) = \sin(\frac{1}{x})$.

Example 15.6 (Squeeze)**.** Let $f(x) = x \sin(\frac{1}{x})$.

Example 15.7 (Steps)**.** For $k \in \mathbb{Z}^+$, let $D_k = (\frac{1}{2^k}, \frac{1}{2^{k-1}})$. Let $\{y_n\}_{n=1}^{\infty}$ be any sequence of real numbers. Consider the function $f : (0, 1] \to \mathbb{R}$ given by $f(x) = y_k$ if $x \in D_k$.

3. One-Sided Limits

Let $D \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Let $f : D \to \mathbb{R}$ and let $L \in \mathbb{R}$.

The *left side of* D *with respect to* x_0 is

$$D_{x_0^-} = (-\infty, x_0) \cap D.$$

We say that x_0 is a *left-sided accumulation point* of D if x_0 is an accumulation point of $D_{x_0^-}$.

The *right side of* D *with respect to* x_0 is

$$D_{x_0^+} = (x_0, \infty) \cap D.$$

We say that x_0 is a *right-sided accumulation point* of D if x_0 is an accumulation point of $D_{x_0^+}$.

Clearly, x_0 is an accumulation point of D if and only if x_0 is either a left-sided or right-sided accumulation point of D, or both. We say that x_0 if a *two-sided* accumulation point of D if it is both a left-sided and a right-sided accumulation point of D.

The *left restriction of* f *with respect to* x_0 is $f \restriction_{D_{x_0^-}}$.

The *right restriction of* f *with respect to* x_0 is $f \restriction_{D_{x_0^+}}$.

We say that L is the *left-sided limit* of f at x_0, and write $L = \lim_{x \to x_0^-} f(x)$, if L is the limit of $f \restriction_{D_{x_0^-}}$.

We say that L is the *right-sided limit* of f at x_0, and write $L = \lim_{x \to x_0^+} f(x)$, if L is the limit of $f \restriction_{D_{x_0^+}}$.

If x_0 is a two-sided accumulation point of D, then L is a limit at x_0 if and only if L is both a left-sided and a right-sided limit at x_0.

4. Limits and Sequences

Lemma 15.8. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence and let $L \in \mathbb{R}$. Then $\{s_n\}_{n=1}^{\infty}$ converges to L if and only if every subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to L.*

Proof. We already saw this. □

Proposition 15.9. *Let $f : D \to \mathbb{R}$ and x_0 an accumulation point of D. Then f has a limit at x_0 if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in $D \setminus \{x_0\}$ converging to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that f has a limit at x_0, and let L be this limit. Let $\epsilon > 0$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $D \setminus \{x_0\}$ which converges to x_0. There exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \epsilon$. Also, since $\{x_n\}_{n=1}^{\infty}$ converges to x_0 and $x_n \neq x_0$ for all n, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $0 < |x_n - x_0| < \delta$. Thus, for $n \geq N$, $|f(x_n) - L| < \epsilon$. Thus $\lim_{n\to\infty} f(x_n) = L$, and in particular, $\{f(x_n)\}_{n=1}^{\infty}$ converges.

($\Leftarrow$) Suppose that for every sequence $\{x_n\}_{n=1}^{\infty}$ in $D \setminus \{x_0\}$ converging to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. We wish to show that f has a limit at x_0.

First we claim that if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequence in $D \setminus \{x_0\}$ which converge to x_0, with limits L_1 and L_2 respectively, then $L_1 = L_2$. To see this, form a new sequence $\{z_n\}_{n=1}^{\infty}$ by $z_{2n-1} = x_n$ and $z_{2n} = y_n$. Then $\{z_n\}_{n=1}^{\infty}$ has a limit, say L. Moreover, every subsequence of $\{z_n\}_{n=1}^{\infty}$ converges to L, and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are subsequences. Thus $L_1 = L_2 = L$.

Now let L denote the common limit of the sequences under consideration; we wish to show that L is the limit of f at x_0. Suppose not. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x with $0 < |x - x_0| < \delta$ but $|f(x) - L| \geq \epsilon$. For $n \in \mathbb{Z}^+$, let $x_n \in \mathbb{R}$ be and element such that $0 < |x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - L| \geq \epsilon$. Then $\{x_n\}_{n=1}^{\infty}$ converges to x_0, but $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. This contradicts the hypothesis. □

Corollary 15.10. *Let $D \subset \mathbb{R}$ and let x_0 be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ have limits at x_0. Then so do $f + g$, fg, and f/g when g is nonzero in $D \cap U$ for some neighborhood U of x_0. Moreover,*

- **(a)** $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$;
- **(b)** $\lim_{x\to x_0}(f(x) \cdot g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$;
- **(c)** $\lim_{x\to x_0}(f(x)/g(x)) = \lim_{x\to x_0} f(x) / \lim_{x\to x_0} g(x)$ *(if appropriate).*

CHAPTER 16

Lecture 16 - Continuity

1. Continuity

Let $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ and let $x_0 \in E$. We say that f is *continuous at* x_0 if

$$\forall \epsilon > 0\, \exists \delta > 0 \ni |x - x_0| \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

If f is continuous at every point in E, we say that f is *continuous* (on E).

The following follows directly from previous definitions and results.

Proposition 16.1. *Let $E \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$ with $x_0 \in E$ and x_0 an accumulation point of E. Then the following are equivalent:*

- **(a)** *f is continuous at x_0;*
- **(b)** *f has a limit at x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$;*
- **(c)** *For every sequence $\{x_n\}_{n=1}^{\infty}$ from E converging to x_0, the sequence $\{f(x_n)\}_{n=0}^{\infty}$ converges to $f(x_0)$.*

Proposition 16.2. *Let $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$. Let $x_0 \in E$. If f and g are continuous at x_0, then so are $f + g$, fg, and f/g (if $g(x_0) \neq 0$).*

Proof. Combine Proposition 16.1 **(c)** with the limit laws for sequences. □

Proposition 16.3. *Let $D, E \subset \mathbb{R}$. Let $f : D \to E$ be continuous at $x_0 \in D$ and $g : E \to \mathbb{R}$ be continuous at $y_0 = f(x_0) \in E$. Then $g \circ f : D \to \mathbb{R}$ is continuous at x_0.*

Proof. Let $\epsilon > 0$. Since g is continuous at y_0, there exists $\delta > 0$ such that

$$|y - y_0| < \delta \Rightarrow |g(y) - g(y_0)| < \epsilon.$$

In particular, for $y = f(x)$ for some x, we rewrite this as

$$|f(x) - f(x_0)| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon.$$

Since f is continuous at x_0, there exists $\gamma > 0$ such that

$$|x - x_0| < \gamma \Rightarrow |f(x) - f(x_0)| < \delta.$$

Then

$$|x - x_0| < \gamma \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon.$$

Thus $g \circ f$ is continuous at x_0. □

2. Continuous Preimage of Open Sets

Remark 16.1. Let $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ and let $x_0 \in E$. Then f is continuous at x_0 if and only if We note that this is identical to the condition

$$\forall \epsilon > 0\, \exists \delta > 0 \ni x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon).$$

Comment. This is simply a rewording of the definition. □

The next proposition implies that the continuous preimage of an open set is open.

Proposition 16.4. *Let $U \subset \mathbb{R}$ be open and let $f : U \to \mathbb{R}$. Then f is continuous on U if and only if for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is open.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that f is continuous on U. Let $V \subset \mathbb{R}$ be open. If $f^{-1}(V) = \varnothing$ it is open, so assume that $f^{-1}(V) \neq \varnothing$, and let $x_0 \in f^{-1}(V)$. We wish to show that a neighborhood of x_0 is contained in V.

Now $f(x_0) \in V$. Since V is open, there exists $\epsilon > 0$ such that $(f(x_0)-\epsilon, f(x_0)+\epsilon) \subset V$. Since f is continuous, there exists $\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \subset V$. So $(x_0 - \delta, x_0 + \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open.

($\Leftarrow$) Suppose that, for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is open. Let $x_0 \in U$ and let $\epsilon > 0$. Let $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$; this is an open set, so $U = f^{-1}(V)$ is open, and $x_0 \in U$, there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset U$. Then $f((x_0-\delta, x_0+\delta)) \subset f(U) \subset V$. Therefore $|x-x_0| < \delta$ implies that $|f(x)-f(x_0)| < \epsilon$, which shows that f is continuous at x_0. □

CHAPTER 17

Lecture 17 - Connected and Compact Sets

1. Goal

We wish to prove the continuous image of a connected set is connected, and that the continuous image of a compact set is compact.

Remark 17.1. Let X and Y be sets and let $f : X \to Y$ be a function. Let $A, B \subset X$ and $C, D \subset Y$. Then

- **(a)** $f^{-1}(f(A)) \supset A$ and equality holds if f is injective;
- **(b)** $f(f^{-1}(C)) \subset C$ and equality holds if f is surjective;
- **(c)** $f(A \cup B) = f(A) \cup f(B)$;
- **(d)** $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
- **(e)** $f(A \cap B) \subset f(A) \cap f(B)$ (give an example where equality fails);
- **(f)** $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

2. Connected Sets

A subset $A \subset \mathbb{R}$ is *disconnected* if there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}$ with $A \cap U_1 \neq \varnothing$ and $A \cap U_2 \neq \varnothing$ such that $A \subset (U_1 \cup U_2)$. Otherwise, we say that A is *connected*.

Proposition 17.1. *Let $A \subset \mathbb{R}$. The following conditions on A are equivalent:*

- **(a)** *there exist $a_1, a_2 \in A$ and $c \notin A$ such that $a_1 < c < a_2$;*
- **(b)** $(\inf(A), \sup(A)) \subset A$*;*
- **(c)** *A is an interval;*
- **(d)** *A is connected.*

Proof. Exercise. □

Proposition 17.2. *Let $f : E \to \mathbb{R}$ be a continuous function, and let $A \subset E$ be connected. Then $f(A)$ is connected.*

Proof. It suffices to show that if $f(A)$ is disconnected, then A is disconnected. Thus assume that $f(A)$ is disconnected, and let V_1 and V_2 be open subsets of $\mathbb{R}$ such that $f(A) \cap V_1 \neq \varnothing$, $f(A) \cap V_2 \neq \varnothing$, but $f(A) \subset (V_1 \cup V_2)$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, but $A \subset (U_1 \cup U_2)$. Moreover, since f is continuous, U_1 and U_2 are open. Thus, A is disconnected. □

3. Compact Sets

Let $A \subset \mathbb{R}$. A *cover* of A is a collection $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ of subsets of $\mathbb{R}$ such that $A \subset \cup\mathcal{C}$.

Let $\mathcal{C}$ be a cover of $A \subset \mathbb{R}$. We say that $\mathcal{C}$ is an *open cover* if every member $U \in \mathcal{C}$ is an open subset of $\mathbb{R}$. We say that $\mathcal{C}$ is a *finite cover* if $\mathcal{C}$ is a finite set.

Note that the modifier *open* refers to the sets inside $\mathcal{C}$, whereas the modifier *finite* refers to the collection $\mathcal{C}$ itself.

A *subcover* of $\mathcal{C}$ is a subset $\mathcal{D} \subset \mathcal{C}$ such that $A \subset \cup\mathcal{D}$.

We say that A is *compact* if every open cover of A has a finite subcover.

Example 17.3. Let $A = \mathbb{Z}$. Let $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{Z}\}$. Then $\mathcal{C}$ is an open cover of $\mathbb{Z}$ with no finite subcover. Thus $\mathbb{Z}$ is not compact.

Example 17.4. Let $A = (0, 1)$. Let $I_n = (0, 1 - \frac{1}{n})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{Z}^+\}$. Then $\mathcal{C}$ is an open cover of $(0, 1)$ with no finite subcover. Thus $(0, 1)$ is not compact.

Proposition 17.5. *Let $A = \{a_1, \dots, a_n\}$ be a finite set. Then A is compact.*

Proof. Let $\mathcal{C}$ be an open cover of A. Then for each $a_i \in A$, there exists and open set $U_i \in \mathcal{C}$ such that $a_i \in U_i$. Then $A \subset \cup_{i=1}^n U_i$, and $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{C}$. Thus A is compact. □

Proposition 17.6. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $[a, b]$.

Let $x \in [a, b]$ and let $U_x \in \mathcal{C}$ be an open set which contains x. Then there exists $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \subset U_x$. Let

$$B = \{x \in [a, b] \mid [a, x] \text{ can be covered by a finite subcover of } \mathcal{C}\}.$$

Note that B is nonempty, since the closed interval $[a, a + \frac{\epsilon_a}{2}] \subset U_a$, and $\{U_a\}$ is a finite subcover of $\mathcal{C}$, so for example $a + \frac{\epsilon_a}{2} \in B$.

Let $z = \sup B$; clearly $a + \frac{\epsilon_a}{2} \leq z \leq b$. We claim that $z \in B$, and that $z = b$. To see this, let $\epsilon = \min\{\epsilon_z, z - a\}$. Then $z - \frac{\epsilon}{2} \in B$. Let $\mathcal{D}$ be a finite subcover of $\mathcal{C}$ which covers $[a, z - \frac{\epsilon}{2}]$. Then $\mathcal{D} \cup \{U_z\}$ covers $[a, z]$, so $z \in B$. Now suppose that $z < b$, and set $\delta = \min\{\epsilon, z - b\}$. Then $z < z + \frac{\delta}{2} < b$, and $\mathcal{D} \cup \{U_z\}$ covers $[a, z + \frac{\delta}{2}]$; since $z + \frac{\delta}{2} \in [a, b]$, this contradicts the definition of z. Thus $z = b$. This completes the proof. □

Proposition 17.7. *Let $f : E \to \mathbb{R}$ be a continuous function, and let $A \subset E$ be a compact set. Then $f(A)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(A)$. Define

$$\mathcal{B} = \{f^{-1}(V) \mid V \in \mathcal{C}\}.$$

Since A is compact, there exists a finite subset of $\mathcal{B}$, say $\mathcal{U} = \{U_1, \dots, U_n\}$, such that $A \subset \cup_{i=1}^n U_i$. Each U_i is the preimage of an open subset, say $U_i = f^{-1}(V_i)$. Then $f(U_i) \subset V_i$, and

$$f(A) \subset f(\cup_{i=1}^n U_i) = \cup_{i=1}^n f(U_i) \subset \cup_{i=1}^n V_i;$$

now $\mathcal{V} = \{V_1, \dots, V_n\}$ is a finite subcover of $\mathcal{C}$. This shows that $f(A)$ is compact. □

Proposition 17.8. *Let $A \subset \mathbb{R}$ be compact and let $F \subset A$ be closed. Then F is compact.*

Proof. Let $\mathcal{C}$ be an open cover of F. Let $U = \mathbb{R} \smallsetminus F$; since F is closed, U is open. Let $\mathcal{B} = \mathcal{C} \cup \{U\}$. Now $\mathcal{B}$ is an open cover of A. Since A is compact, let $\mathcal{U}$ be a finite subcover of A. Since $F \subset A$, then $\mathcal{U}$ is also a finite open cover of F. Let $\mathcal{V} = \mathcal{U} \smallsetminus \{U\}$; now $\mathcal{V}$ is still a finite open cover of F, and $\mathcal{V}$ is a subcover of $\mathcal{C}$. Thus F is compact. □

Theorem 17.9 (Heine-Borel Theorem)**.** *Let $A \subset \mathbb{R}$. Then A is compact if and only if A is closed and bounded.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that A is compact; we wish to show that A is closed and bounded.

Cover A with sets of the form $(-n, n)$, for $n \in \mathbb{Z}^+$. Since A is compact, there exists a finite subcover. This subcover contains an interval of maximum length, say $(-M, M)$, and clearly $A \subset (-M, M)$. Thus $A \subset [-M, M]$, and A is bounded.

To show that A is closed, we show that its complement is open. Let $B = \mathbb{R} \smallsetminus A$. Let $b \in B$. For each point $a \in A$, set $\epsilon_a = |b - a|/2$, $I_a = (a - \epsilon, a + \epsilon)$, and $J_a = (b - \epsilon, b + \epsilon)$. Let $\mathcal{I} = \{I_a \mid a \in A\}$. Then $\mathcal{I}$ is an open cover of A, and so it has a finite subcover $\{I_{a_1}, \dots, I_{a_n}\}$. The open set $\cup_{i=1}^n I_{a_i}$ contains A and is disjoint from the set $\cap_{i=1}^n J_{a_i}$, which is also open and contains b. Thus B is open.

($\Leftarrow$) Suppose that A is closed and bounded; we wish to show that A is compact. Since A is bounded, there exists $M > 0$ such that $A \subset [-M, M]$. The set $[-M, M]$ is a closed interval, and so it is compact by Proposition 17.6. Thus A is a closed subset of a compact set, and therefore is compact by Proposition 17.8. □

Proposition 17.10. *Let K be a compact set. Then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. Since K is bounded, then $\sup K$ exists as a real number, say $b = \sup K$. Suppose $b \notin K$; then $\{(-\infty, b - \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$ is an open cover of K with no finite subcover, contradicting that K is compact. Thus $b \in K$. Similarly, $\inf K \in K$. □

CHAPTER 18

Lecture 18 - Uniform Continuity

1. Boundary Behavior

Recall that if $f : D \to \mathbb{R}$ is continuous and $x_0 \in D$ is an accumulation point of D, then f has a limit at x_0, and indeed $\lim_{x \to x_0} f(x) = f(x_0)$. However, if x_0 is not in D, other possibilities exist.

Let $D = (-1, 1)$ and $x_0 = 1$; clearly, x_0 is an accumulation point of D. Let $f : D \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1-x^2}$. Now f is continuous on D and x_0 is an accumulation point of D, but the limit does not exist at x_0.

This happens because continuity is a *local property*, as opposed to a *global property*. A local property is one which does not look beyond some neighborhood of every point, no matter how small that neighborhood may be.

For example, whether or not a set is open is a local property of the set, but whether or not it is bounded is a global property.

2. Uniform Continuity

Let $E \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$. We say that f is *uniformly continuous* on E if

$$\forall \epsilon > 0 \, \exists \delta > 0 \ni |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon,$$

where $x, y \in E$.

Proposition 18.1. *Let $f : D \to \mathbb{R}$ be uniformly continuous and let x_0 be an accumulation point of D. Then f has a limit at x_0.*

Proof. Recall that f has a limit at x_0 if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in D which converges to x_0, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in D which converges to x_0; to show that $\{f(x_n)\}_{n=1}^{\infty}$ converges, it suffices to show that it is a Cauchy sequence.

Let $\epsilon > 0$; we wish to find $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|f(x_m) - f(x_n)| < \epsilon$. Since f is uniformly continuous, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$, where $x, y \in E$.

Since $\lim_{n \to \infty} x_n = x_0$, there exists $N \in \mathbb{Z}^+$ such that $|x_n - x_0| < \frac{\delta}{2}$ whenever $n \geq N$. Then for $m, n \geq N$, we have $|x_m - x_n| < \delta$, so $|f(x_m) - f(x_n)| < \epsilon$. Since ϵ was selected arbitrarily, this shows that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, and is therefore convergent. □

Proposition 18.2. *Let $f : E \to \mathbb{R}$ be a continuous function. If E is compact, then f is uniformly continuous.*

Proof. Suppose that E is compact.

Let $\epsilon > 0$; we wish to find $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Since f is continuous, then for every $x \in E$ there exists $\delta_x > 0$ such that if $y \in E$ and $|x - y| < \delta_x$, then $|f(x) - f(y)| < \epsilon$.

Let $V_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$; this is an open set which contains x. Let $\mathcal{C} = \{V_x \mid x \in E\}$; then $\mathcal{C}$ is an open cover of E. Since E is compact, there exists a finite subcover, so there exist $x_1, \ldots, x_n \in E$ such that $E \subset \cup_{i=1}^{n} V_{x_n}$. Set

$$\delta = \min\{\delta_{x_i}/2 \mid i = 1, \ldots, n\}/2.$$

Let $x, y \in E$ with $|x - y| < \delta$. Now there exists x_i such that $|x - x_i| < \frac{\delta_{x_i}}{2}$. Then

$$\begin{aligned} |y - x_i| &= |y - x + x - x_i| \\ &\leq |y - x| + |x - x_i| \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\ &= \delta. \end{aligned}$$

□

CHAPTER 19

Lecture 19 - Intermediate Value Theorem

1. Relatively Open Sets

Let $E \subset \mathbb{R}$ and let $V \subset E$. We say that E is *relatively open* in E if there exists an open set $U \subset \mathbb{R}$ such that $U \cap E = V$. Similarly, a subset $G \subset E$ is *relatively closed* if $E \setminus G$ is relatively open. This is equivalent to the existence of a closed set $F \subset \mathbb{R}$ such that $F \cap E = G$.

Proposition 19.1. *Let $f : E \to \mathbb{R}$ be a function. Then f is continuous on E if and only if for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is relatively open in E.*

2. Homeomorphism

Let A and B be subsets of $\mathbb{R}$. A *homeomorphism* from A to B is a bijective continuous function $f : A \to B$ such that f^{-1} is also continuous.

It is natural to suppose that any bijective continuous function is a homeomorphism, but this is not the case.

Example 19.2. Let $A = (0, 1) \cup [2, 3)$ and let $B = (0, 2)$. Define $f : A \to B$ by

$$f(x) = \begin{cases} x & \text{if } x \in (0,1); \\ x - 1 & \text{if } x \in [2,3). \end{cases}$$

This function is clearly bijective and continuous at every point in A; however, its inverse is discontinuous.

We have seen that the continuous image of a compact set is compact. We will use this fact in the next proposition.

Proposition 19.3. *Let $f : A \to B$ be a bijective continuous function. If A is compact, then f is a homeomorphism.*

Lemma 19.4. *If F is closed and U is open, then $F \setminus U$ is closed and $U \setminus F$ is open.*

Proof of Lemma. Since $F \setminus U = F \cap (\mathbb{R} \setminus U)$ is the intersection of closed sets, it is closed. On the other hand, since $U \setminus F = U \cap (\mathbb{R} \setminus F)$ is the intersection of open sets, it is open. □

Proof of Proposition. Let $g = f^{-1}$ so that $g : B \to A$ is a bijective function; we wish to show that g is continuous.

Let $\epsilon > 0$ and select $x_0 \in B$. Since A is compact, it is closed and bounded Let $U = (g(x_0) - \epsilon, g(x_0) + \epsilon)$. Then U is open, and $K = A \setminus U$ is also closed and bounded, and hence compact. Since the continuous image of a compact set is compact, we see that $f(K)$ is compact, and hence closed. Let $V = \mathbb{R} \setminus f(K)$; this set is open. Note that

$$\begin{aligned} g(B \cap V) &= g(B \setminus f(K)) \\ &= g(B) \setminus g(f(K)) \\ &= g(B) \setminus K \\ &= A \setminus (A \setminus U) \\ &= U. \end{aligned}$$

Now $g(x_0) \notin K$, so $x_0 = f(g(x_0)) \notin f(K)$, so $x_0 \in V$. Therefore there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset V$. Thus if $x \in B$ and $|x - x_0| < \delta$, we have $f(x) \in U$, which says that $|f(x) - f(x_0)| < \epsilon$. □

3. Connectedness Revisited

Recall the definition of a closed interval:

$$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}.$$

Recall the definition of connectedness:

A subset $A \subset \mathbb{R}$ is *disconnected* if there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}$ with $A \cap U_1 \neq \varnothing$ and $A \cap U_2 \neq \varnothing$ such that $A \subset (U_1 \cup U_2)$. Otherwise, we say that A is *connected*.

Proposition 19.5. *Let $f : E \to \mathbb{R}$ be a continuous function, If E is connected, then $f(E)$ is connected.*

Proof. It suffices to show that if $f(E)$ is disconnected, then E is disconnected. Thus assume that $f(E)$ is disconnected, and let V_1 and V_2 be open subsets of $\mathbb{R}$ such that $f(E) \cap V_1 \neq \varnothing$, $f(E) \cap V_2 \neq \varnothing$, but $f(E) \subset (V_1 \cup V_2)$.

Let $E_1 = f^{-1}(V_1)$ and $E_2 = f^{-1}(V_2)$. We wish to find disjoint open sets U_1 and U_2 such that $E_1 = E \cap U_1$ and $E_2 = E \cap U_2$.

For each $y \in f(E)$ there exists $\epsilon_y > 0$ such that $(y - \epsilon_y, y + \epsilon_y) \subset V_i$, where $y \in V_i$. Since f is continuous, for each $x \in E$ there exists $\delta_x > 0$ such that $f((x - \delta_x, x + \delta_x)) \subset (y - \epsilon_y, y + \epsilon_y)$, where $y = f(x)$.

Set $U_i = \cup_{x \in E_i} (x - \delta_x, x + \delta_x)$, for $i = 1, 2$. Then U_1 and U_2 are open sets. Also $E \cap U_1 \neq \varnothing$, $E \cap U_2 \neq \varnothing$, but $E \subset (U_1 \cup U_2)$. Thus, E is disconnected. □

Proposition 19.6. *Let $A \subset \mathbb{R}$. Then A is connected if and only if*

$$a, b \in A \Rightarrow [a,b] \subset A.$$

Proof. We prove both directions.

($\Rightarrow$) Let $a, b \in A$ with $a < b$ and suppose that $[a,b]$ is not contained in A. Then there exists $c \in [a,b]$ such that $c \notin A$. Set $U_1 = (-\infty, c)$ and $U_2 = (c, \infty)$; then $a \in U_1$, $b \in U_2$, and $A \subset U_1 \cup U_2$. Thus A is disconnected.

($\Leftarrow$) Suppose that for every $a, b \in A$ with $a < b$, we have $[a,b] \subset A$. Let U_1 and U_2 be open sets with $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. We wish to show that $U_1 \cap U_2 \neq \varnothing$.

Let $a \in U_1$ and $b \in U_2$; without loss of generality, assume that $a < b$. Let $c = \sup U_1 \cap [a,b]$. Clearly $c \in [a,b]$, so either $c \in U_1$ or $c \in U_2$.

If $c \in U_1$, then there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_1$. Thus $c + \min\{\frac{\epsilon}{2}, \frac{c+b}{2}\}$ is also in U_1 and in $[a,b]$, contradicting the definition of c.

Thus $c \in U_2$, so there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_2$. But by the definition of c, there exists $d \in U_1 \cap [a,b]$ such that $d \in (c - \epsilon, c) \subset U_2$. Thus $U_1 \cap U_2 \neq \varnothing$. □

Proposition 19.7. *Let $K \subset \mathbb{R}$ be a compact set. Then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. Exercise. □

Proposition 19.8. *A compact connected set is a closed interval.*

Proof. Exercise. □

4. Intermediate Value Theorem

Theorem 19.9. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function with* $f(a) \cdot f(b) < 0$. *Then there exists* $c \in [a, b]$ *such that* $f(c) = 0$.

Proof. Assume that $f(a) < 0 < f(b)$. Since $f([a, b])$ is connected, $[f(a), f(b)] \subset f([a, b])$. Since $0 \in [f(a), f(b)]$, then $0 \in f([a, b])$. That is, $f(c) = 0$ for some $c \in f([a, b])$. □

CHAPTER 20

Lecture 20 - Differentiation

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in D$ be an accumulation point of D. Define a function

$$\widehat{f}_a : D \smallsetminus \{a\} \to \mathbb{R} \quad \text{by} \quad \widehat{f}_a(x) = \frac{f(x) - f(a)}{x - a}.$$

We say that f is *differentiable* at a if $\widehat{f}_a$ has a limit at a. In this case, we write

$$f'(a) = \lim_{x \to a} \widehat{f}_a(x),$$

and we call $f'(a)$ the *derivative* of f at a.

Proposition 20.1. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$. If f is differentiable at a, then f is continuous at a.*

Proof. Suppose that f is differentiable at a. Then $\widehat{f}_a$ has a limit at a. Now for $x \in D \smallsetminus \{a\}$, we have

$$f(x) = \widehat{f}_a(x)(x - a) + f(a).$$

The constituent functions of the right hand side have limits; thus the left hand side has a limit, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} \widehat{f}_a(x) \lim_{x \to a}(x - a) + \lim_{x \to a} f(a) = f'(a) \cdot 0 + f(a) = f(a).$$

Thus f is continuous at a. □

Proposition 20.2. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be differentiable at a. Then $f + g$ is differentiable at a, and $(f + g)'(a) = f'(a) + g'(a)$.*

Proof. Notice that

$$\begin{aligned}
(\widehat{f + g})_a(x) &= \frac{(f + g)(x) - (f + g)(a)}{x - a} \\
&= \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a} \\
&= \widehat{f}_a(x) + \widehat{g}_a(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{x \to a} (\widehat{f + g})_a(x) &= \lim_{x \to a} \widehat{f}_a(x) + \lim_{x \to a} \widehat{g}_a(x) \\
&= f'(a) + g'(a).
\end{aligned}$$

□

Proposition 20.3. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ be differentiable at a and let $c \in \mathbb{R}$. Then cf is differentiable at a, and $(cf)'(a) = cf'(a)$.*

Proposition 20.4. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be differentiable at a. Then fg is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*

Proposition 20.5. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be differentiable at a, with $g(a) \neq 0$. Then $\frac{f}{g}$ is differentiable at a, and $(\frac{f}{g})'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$.*

CHAPTER 21

Lecture 21 - Properties of the Derivative

1. Leibnitz Notation

Let $f : D \to \mathbb{R}$ with x_0 an accumulation point of D.

Let $\Delta x = x - x_0$; viewing x_0 as fixed, this is implicitly a function of x. Let $\Delta f = f(x) - f(x_0)$; viewing f as fixed, this is also a function of x.

Now x goes to x_0, we see that Δx goes to 0. Thus

$$\lim_{x\to 0} \frac{\Delta f}{\Delta x} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Thus we may define the derivative to be

$$\frac{df}{dx} = \lim_{x\to 0} \frac{\Delta f}{\Delta x}.$$

Moreover, in Leibnitz notation, it is traditional to start with a function whose name is y instead of f, so this becomes

$$\frac{dy}{dx} = \lim_{x\to 0} \frac{\Delta y}{\Delta x}.$$

2. Chain Rule

If we have two lines $y = m_1 z + b_1$ and $z = m_2 x + b_2$ and compose them, we obtain

$$y = m_1 m_2 x + (m_1 b_2 + b_1),$$

a line with slope $m_1 m_2$. Since we view a differentiable function as a function which is approximately a line whose slope is the derivative, we guess that the derivative of a composition is the product of the derivatives.

Suppose that y is a function of u and u is a function of x. Then we may attempt to right

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Then $\Delta u \to 0$ as $\Delta x \to 0$, so taking the limit of both sides we would arrive at

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The problem with this reasoning is that Δu may be zero even when Δx is nonzero. We have to get around this problem.

Proposition 21.1 (Chain Rule)**.** *Let $X, Y \subset \mathbb{R}$ with $x_0 \in D$ an accumulation point of X and $y_0 \in Y$ and accumulation point of Y. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ with $f(X) \subset Y$ and $f(x_0) = y_0$. If f is differentiable at x_0 and g is differentiable at y_0, then $g \circ f$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0).$$

Proof. Define a function $U : X \to \mathbb{R}$ by $U(x) = \frac{g(f(x)) - g(f(x_0))}{x - x_0}$; we wish to show that $U(x)$ has a limit at $x = x_0$, and that $\lim_{x \to x_0} U(x) = g'(y_0) f'(x_0)$.

Define $h : Y \to \mathbb{R}$ by

$$h(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} & \text{if } y \neq y_0; \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Since g is differentiable at y_0, we have $\lim_{y \to y_0} h(y) = g'(y_0) = h(y_0)$, so h is continuous at y_0. Since f is differentiable at x_0, it is continuous at x_0, and since $f(x_0) = y_0$, then $h \circ f$ is continuous at x_0.

Set $T(x) = \frac{f(x) - f(x_0)}{x - x_0}$. We claim that for $x \in D \setminus \{x_0\}$, we have $U(x) = h(f(x)) \cdot T(x)$. If $f(x) = f(x_0)$, then $g(f(x)) = g(f(x_0)) = g(y_0)$. In this case, $U(x) = 0$ and $h(f(x)) \cdot T(x) = 0$. Otherwise, $U(x) = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} = h(f(x)) \cdot T(x)$.

Now take the limit to see that

$$\lim_{x \to x_0} U(x) = \lim_{x \to x_0} h(f(x)) \lim_{x \to x_0} T(x) = g'(y_0) f'(x_0).$$

∎

3. Extrema

Let $f : D \to \mathbb{R}$ and let $x_0 \in D$.

We call x_0 a *global maximum* [respectively *global minimum*] of f if $f(x) \le f(x_0)$ [respectively $f(x) \ge f(x_0)$] for all $x \in [a, b]$. If x_0 is a global minimum or a global maximum, it is called a *global extremum*.

Proposition 21.2. *Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$ be continuous. If D is compact, then there exist $x_1, x_2 \in D$ such that x_1 is a global minimum of f and x_2 is a global maximum of f.*

Proof. Since D is compact and f is continuous, then $f(D)$ is compact. Thus $\inf f(D) \in f(D)$ and $\sup f(D) \in f(D)$. So there exist $x_1, x_2 \in D$ such that $f(x_1) = \inf f(D)$ and $f(x_2) = \sup f(D)$. Then x_1 is a global minimum and x_2 is a global maximum. □

We call x_0 a *local maximum* [respectively *local minimum*] of f if there exists a neighborhood Q of x_0 such that for $x \in Q \cap D$ we have $f(x) \le f(x_0)$ [respectively $f(x) \ge f(x_0)$]. If x_0 is a local minimum or a local maximum, it is called a *local extremum*.

Proposition 21.3. *Let $f : [a, b] \to \mathbb{R}$ and let $x_0 \in [a, b]$ be a local extremum of f. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose that x_0 is a local maximum. Then there exists $\delta > 0$ such that $f(x) \le f(x_0)$ for all x satisfying $|x - x_0| < \delta$.

Set $T(x) = \frac{f(x)-f(x_0)}{x-x_0}$ for $x \in D \setminus \{x_0\}$. Since f is differentiable at x_0, $\lim_{n\to\infty} T(x_n) = f'(x_0)$ for every sequence from $(x - \delta, x + \delta)$ which converges to x_0.

Note that the numerator of $T(x)$ is negative for x near x_0. For $x_n = x - \frac{\delta}{n}$, we see that $T(x_n) \ge 0$, so $f'(x_0) \ge 0$. However, for $x_n = x + \frac{\delta}{n}$, we have $T(x_n) \le 0$, so $f'(x_0) \le 0$. This shows that $f'(x_0) = 0$. □

4. Rolle's Theorem

Proposition 21.4 (Rolle's Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. Then if* $f(a) = f(b)$*, there exists* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

Proof. Since $[a, b]$ is compact, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1)$ is a global minimum and $f(x_2)$ is a global maximum. If $f(x_1) = f(x_2)$, then f is constant, and $f'(x) = 0$ for every $x \in [a, b]$. Otherwise, either $x_1 \neq f(a)$ or $x_2 \neq f(a)$. Therefore either $x_1 \in (a, b)$ or $x_2 \in (a, b)$. If $x_1 \in (a, b)$, then x_1 is a local minimum, and $f'(x_1) = 0$. If $x_2 \in (a, b)$, then x_2 is a local maximum, and $f'(x_2) = 0$. □

Proposition 21.5 (Mean Value Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. Then there exists* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(b)}{b-a}$*.*

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - \frac{x-a}{b-a}(f(b) - f(a))$. There g is continuous on $[a, b]$ and differentiable on (a, b), and we compute that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

However, $g(a) = f(a)$ and $g(b) = f(a)$. By Rolle's Theorem, there exists $c \in [a, b]$ such that $g'(c) = 0$, so $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

5. Inverse Function Theorem

Proposition 21.6. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) \neq 0$ for $x \in [a,b]$, then f is injective, and the inverse of f is differentiable on $f((a,b))$, with*

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

for every $x \in (a,b)$.

Proof. Suppose that f is not injective. Then there exist $x_1, x_2 \in [a,b]$ such that $f(x_1) = f(x_2)$. By Rolle's Theorem, there exists $c \in [x_1, x_2]$ such that $f'(c) = 0$; this violates the hypothesis. Thus f is injective.

We have seen that a continuous bijective function on a compact set has a continuous inverse; since $[a,b]$ is compact, $f^{-1} : f([a,b]) \to [a,b]$ is continuous.

Now let $y_0 \in f((a,b))$, and let $\{y_n\}_{n=1}^{\infty}$ be an arbitrary sequence from $f((a,b)) \setminus \{y_0\}$ which converges to y_0. Set $x_0 = f^{-1}(y_0)$ and $x_n = f^{-1}(y_n)$. It suffices to show that $\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$.

Since f^{-1} is continuous, we see that $\lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0)$, that is, $\lim_{n\to\infty} x_n = x_0$. Thus, since f is differentiable at x_0, we have

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0).$$

Since f is injective, $f(x_n) - f(x_0) \neq 0$ unless $x_n = x_0$, so by a property of limits of sequences we have

$$\frac{1}{f'(x_0)} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}.$$

□

CHAPTER 22

Lecture 22 - Integration

Let $a, b \in \mathbb{R}$ with $a < b$. A *partition* of $[a, b]$ is a finite set $\{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$.

Let $\mathcal{N}(a, b)$ denote the set of all partitions of $[a, b]$. Note that this set is partially ordered by inclusion. If $P, Q \in \mathcal{N}(a, b)$ and $P \subset Q$, we say that Q is a *refinement* of P.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Set

$$M_f(P, i) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \quad \text{and} \quad m_f(P, i) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

The *upper Riemann sum* of f with respect P is

$$U_f(P) = \sum_{i=1}^{n} M_f(P, i)(x_i - x_{i-1}),$$

and the *lower Riemann sum* of f with respect to P is

$$L_f(P) = \sum_{i=1}^{n} m_f(P, i)(x_i - x_{i-1}).$$

Since f is bounded, there exist $m, M \in \mathbb{R}$ with $m < M$ such that $f(x) \in [m, M]$ for every $x \in [a, b]$. Thus

$$m(b - a) \leq L_f(P) \leq U_f(P) \leq M(b - a).$$

Moreover, if Q is a refinement of P, then

$$L_f(P) \leq L_f(Q) \leq U_f(Q) \leq U_f(P).$$

The *upper Riemann integral* of f is

$$\overline{\int_a^b} f\,dx = \inf\{U_f(P) \mid P \in \mathcal{N}(a, b)\},$$

and the *lower Riemann integral* of f is

$$\underline{\int_a^b} f\,dx = \sup\{L_f(P) \mid P \in \mathcal{N}(a, b)\}.$$

Note that the upper and lower Riemann integrals exist for any bounded function; in fact, if $f(x) \in [m, M]$ for every $x \in [a, b]$, then

$$m(b - a) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq M(b - a).$$

We say that f is *Riemann integrable* on $[a, b]$ if $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$. The common value is called the *Riemann integral*, and is denoted by $\int_a^b f\,dx$.

The adjective *Riemann* precedes the word *integrable* because there are other sorts of integrals which are of as much or more importance to theoretical mathematics as the Riemann integral. Predominant among these is the *Lebesque* integral, which is defined by splitting up the range of the function f instead of its domain. However, we will not study Lebesque integrable functions, and the modifier Riemann becomes superfluous for us. Thus we will call a Riemann integrable function simply *integrable*.

Proposition 22.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the following conditions are equivalent:*

- **(a)** *f is integrable on $[a, b]$;*
- **(b)** $\overline{\int}_a^b f\,dx - \underline{\int}_a^b f\,dx = 0$*;*
- **(c)** $\inf\{U_f(P) - L_f(P) \mid P \in \mathcal{N}(a, b)\} = 0$*;*
- **(d)** $\forall \epsilon > 0\, \exists P \in \mathcal{N}(a, b) \ni U_f(P) - L_f(P) < \epsilon$*.*

Proof. It is obvious that **(a)** is equivalent to **(b)**. Also, that **(c)** implies **(d)** is clear. That **(d)** implies **(c)** follows immediately from the fact that $U_f(P) \geq L_f(P)$ for every $P \in \mathcal{N}(a, b)$.

Suppose that f is integrable on $[a, b]$, and set $I = \int_a^b f\,dx$, that is,

$$\sup\{L_f(P) \mid P \in \mathcal{N}(a, b)\} = I = \inf\{U_f(P) \mid P \in \mathcal{N}(a, b)\}.$$

Let $\epsilon > 0$. Then there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U_f(P_1) - \frac{\epsilon}{2} < I < L_f(P_2) + \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$; then P is a common refinement of P_1 and P_2, and

$$U_f(P) - \frac{\epsilon}{2} \leq U_f(P_1) - \frac{\epsilon}{2} < I < L_f(P_2) + \frac{\epsilon}{2} \leq L_f(P) + \frac{\epsilon}{2}.$$

This implies that

$$U_f(P) - L_f(P) < \epsilon,$$

which shows that **(a)** implies **(d)**.

Now suppose that condition **(d)** holds. Let $\epsilon > 0$, and let $P \in \mathcal{N}(a, b)$ such that $U_f(P) - L_f(P) < \epsilon$. Now $\overline{\int}_a^b f\,dx \leq U_f(P)$, and $\underline{\int}_a^b f\,dx \geq L_f(P)$. Subtracting these inequalities yields

$$0 \leq \overline{\int}_a^b f\,dx - \underline{\int}_a^b f\,dx \leq U_f(P) - L_f(P) < \epsilon.$$

Since ϵ is arbitrary, this proves **(b)**. $\square$

Example 22.2. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0,1]$. Then

$$U_f(P) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \quad \text{and} \quad L_f(P) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})$$

Then

$$U_f(P) - L_f(P) = \sum_{i=1}^{n} \left[x_i(x_i - x_{i-1}) - x_{i-1}(x_i - x_{i-1})\right] = \sum_{i=1}^{n} (x_i - x_{i-1})^2.$$

Let $\epsilon > 0$ and let n be so large that $n > \frac{1}{\epsilon}$. Define a partition P by $P = \{\frac{k}{n} \mid k = 0, \ldots, n\}$. Then

$$U_f(P) - L_f(P) = \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n} < \epsilon.$$

By the previous proposition, f is integrable.

Example 22.3. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0,1]$. Then for every i, $M_f(P,i) = 1$ and $m_f(P,i) = 0$, so $U_f(P) = 1$ and $L_f(P) = 0$. Therefore $\overline{\int_0^1} f\,dx = 1$ and $\underline{\int_0^1} f\,dx = 0$, so f is not integrable.

Example 22.4. Define $q : \mathbb{Q} \to \mathbb{Z}^+$ by

$$q(x) = \min\{b \in \mathbb{Z}^+ \mid x = \frac{a}{b} \text{ for some } a \in \mathbb{Z}\}.$$

Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{q(x)} & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Since every interval contains an irrational number, it is clear that $\underline{\int}_0^1 f\,dx = 0$.

Therefore, if f is integrable, we would have $\int_a^b f\,dx = 0$. We wish to show that the upper Riemann integral is zero.

Let $\epsilon > 0$. We construct a partition P of $[0,1]$ such that $U_f(P) < \epsilon$.

There are only finitely many rational numbers $t \in (0,1)$ such that $\frac{1}{q(t)} \geq \frac{\epsilon}{2}$; let $\{t_1, \ldots, t_m\}$ be the set of such numbers, with $t_i < t_{i+1}$. Set

$$h = (\min\{t_{i+1} - t_i\} \cup \{\frac{\epsilon}{m}, 1 - t_m\})/2.$$

Then the intervals of the form $[t_i, t_i + h]$ are disjoint. Set $x_0 = 0$ and $x_{2m+1} = 1$, and for $i = 1, \ldots, m$, set $x_{2i-1} = t_i$ and $x_{2i} = t_i + h$.

Set $n = 2m+1$. Now $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[0,1]$. For i odd, then $M_f(P,i) < \frac{\epsilon}{2}$. For i even, then $(x_i - x_{i-1}) \leq \frac{\epsilon}{2m}$. Thus

$$\begin{aligned} U_f(P) &= \sum_{i=1}^{n} M_f(P,i)(x_i - x_{i-1}) \\ &= \sum_{\text{odd}} M_f(P,i)(x_i - x_{i-1}) + \sum_{\text{even}} M_f(P,i)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} \sum_{\text{odd}} (x_i - x_{i-1}) + h \sum_{\text{even}} M_f(P,i) \\ &< \frac{\epsilon}{2} + hm \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$. We say that f is *increasing* on $[a, b]$ if for every $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Proposition 22.5. *Let $f : [a, b] \to \mathbb{R}$ be increasing. Then f is integrable on $[a, b]$.*

Proof. Since f is increasing, $f(x) \in [f(a), f(b)]$ for every $x \in [a, b]$. In particular, f is bounded. Set $B = f(b) - f(a)$.

Let $P = \{x_0, \dots, x_n\}$ be any partition of $[a, b]$. Since f is increasing, we have $M_f(p, i) = \max\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_i)$, and $m_f(P, I) = \min\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_{i-1})$. Then

$$U_f(P) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \quad \text{and} \quad L_f(P) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}),$$

so $U_f(P) - L_f(P) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$.

Let $\epsilon > 0$, and let $k = \frac{\epsilon}{2B}$ so that $0 < kB < \epsilon$. Choose a partition $P = \{x_0, x_1, \dots, x_n\}$ such that $x_i - x_{i-1} < k$. Then

$$\begin{aligned} U_f(P) - L_f(P) &\leq \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))k \\ &= k \sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) \\ &= kM < \epsilon. \end{aligned}$$

Thus f is integrable. □

Proposition 22.6. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a, b]$.*

Proof. Let $\epsilon > 0$; we wish to find a partition P such that $U_f(P) - L_f(P) < \epsilon$).

Since f is continuous and $[a, b]$ is compact, the image is also compact, and in particular, f is bounded on $[a, b]$. Moreover, f is uniformly continuous on $[a, b]$, so there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$ such that $|x_i - x_{i-1}| < \delta$. There exist $s_i, t_i \in [x_{i-1}, x_i]$ such that $f(s_i) = m_f(P, i)$ and $f(t_i) = M_f(P, i)$. Since $|s_i - t_i| < \delta$, we have $|f(t_i) - f(s_i)| < \epsilon/(b - a)$. Thus

$$\begin{aligned} U_f(P) - L_f(P) &= \sum_{i=1}^{n}(f(t_i) - f(s_i))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= \epsilon. \end{aligned}$$

Thus f is integrable. □

CHAPTER 23

Lecture 23 - Integration Properties

Proposition 23.1. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $(f+g) : [a,b] \to \mathbb{R}$ is integrable, and*

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx.$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Then

$$m_f(P,i) + m_g(P,i) \leq m_{f+g}(P,i) \leq M_{f+g}(P,i) \leq M_f(P,i) + M_g(P,i),$$

for $i = 1, \ldots, n$. Therefore

$$L_f(P) + L_g(P) \leq L_{f+g}(P) \leq U_{f+g}(P) \leq U_f(P) + U_g(P). \tag{1}$$

Next we would like to say this: since this is true for every partition P, we have

$$\underline{\int_a^b} f\,dx + \underline{\int_a^b} g\,dx \leq \underline{\int_a^b} (f+g)\,dx \leq \overline{\int_a^b} (f+g)\,dx \leq \overline{\int_a^b} f\,dx + \overline{\int_a^b} g\,dx.$$

However, this path is actually more difficult to justify than it first appears. It is easier to proceed as follows:

Inequality (1) implies that

$$(U_f(P) - L_f(P)) + (U_g(P) - L_g(P)) \geq U_{f+g}(P) - L_{f+g}(P) \geq 0.$$

Let $\epsilon > 0$; then there exists a partition P_1 such that $U_f(P) - L_f(P) < \frac{\epsilon}{2}$, and there exists a partition P_2 such that $U_g(P) - L_g(P) < \frac{\epsilon}{2}$. Let $P = P_1 \cup P_2$; then

$$U_{f+g}(P) - L_{f+g}(P) \leq (U_f(P) - L_f(P)) + (U_g(P) - L_g(P)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f+g$ is integrable. Moreover, $\int_a^b f\,dx - \frac{\epsilon}{2} < L_f(P)$, $\int_a^b g\,dx - \frac{\epsilon}{2} < L_g(P)$, $\int_a^b f\,dx - \frac{\epsilon}{2} > U_f(P)$, and $\int_a^b g\,dx - \frac{\epsilon}{2} > U_g(P)$; therefore

$$\int_a^b f\,dx + \int_a^b g\,dx - \epsilon \leq \int_a^b (f_g)\,dx \leq \int_a^b f\,dx + \int_a^b g\,dx + \epsilon.$$

Since this is true for every ϵ, we must have

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx.$$

□

Proposition 23.2. *Let $f : [a, b] \to \mathbb{R}$ be integrable and let $c \in \mathbb{R}$ Then $cf : [a, b] \to \mathbb{R}$ is integrable, and*

$$\int_a^b cf\,dx = c\int_a^b f\,dx.$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Then

$$cm_f(P, i) = m_{cf}(P, i) \leq M_{cf}(P, i) = cM_f(P, i),$$

for $i = 1, \ldots, n$. Thus

$$cL_f(P) = L_{cf}(P) \leq U_{cf}(P) = cU_f(P).$$

Assume $c \geq 0$. Then for any bounded set X, we have $c \sup X = \sup\{cx \mid x \in X\}$. This gives

$$c\underline{\int_a^b} f\,dx = \underline{\int_a^b} cf\,dx \leq \overline{\int} cf\,dx = c\overline{\int} f\,dx;$$

since f is integrable, the result follows in this case.

The case of $c = -1$ follows from the fact that $-m_f(P, i) = M_{-f}(P, i)$, which we leave as an exercise. □

Proposition 23.3. *Let $f : [a, b] \to \mathbb{R}$, and let $c \in [a, b]$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and on $[c, b]$, in which case we have*

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx.$$

Proof. Suppose that f is integrable on $[a, c]$ and on $[c, b]$, and let $\epsilon > 0$. Then there exist partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U_f(P_1) - \frac{\epsilon}{4} < \int_a^c f\,dx < L_f(P_1) + \frac{\epsilon}{4},$$

and

$$U_f(P_2) - \frac{\epsilon}{4} < \int_c^b f\,dx < L_f(P_2) + \frac{\epsilon}{4}.$$

Let $P = P_1 \cup P_2$; this is a partition of $[a, b]$. Adding these inequalities yields

$$U_f(P) - \frac{\epsilon}{2} < \int_a^c f\,dx + \int_c^b f\,dx < L_f(P) + \frac{\epsilon}{2}.$$

Therefore $U_f(P) - L_f(P) < \epsilon$, so f is integrable on $[a, b]$, and the above inequality implies that

$$\int_a^b f\,dx - \frac{\epsilon}{2} < \int_a^c f\,dx + \int_c^b f\,dx < \int_a^b f\,dx + \frac{\epsilon}{2}.$$

Since this is true for every ϵ, we have

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx.$$

Suppose that f is integrable on $[a, b]$, and let $\epsilon > 0$. Then there exists a partition $P = \{x_0, \ldots, x_n\}$ such that $U_f(P) - L_f(P) < \epsilon$, and we may assume that $c \in P$, so that $c = x_k$ for some k. Then $P_1 = \{x_0, \ldots, x_k\}$ is a partition of $[a, c]$, and $P_2 = \{x_k, \ldots, x_n\}$ is a partition of $[c, b]$.

Clearly $U_f(P) = U_f(P_1) + U_f(P_2)$ and $L_f(P) = L_f(P_1) + L_f(P_2)$. Then

$$(U_f(P_1) - L_f(P_1)) + (U_f(P_2) - L_f(P_2)) < \epsilon.$$

Since each summand is positive, each is less than epsilon, which proves the f is integrable on $[a, c]$ and on $[c, b]$. $\square$

CHAPTER 24

Lecture 24 - Fundamental Theorem of Calculus

Proposition 24.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that $f' : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$. Then*

$$\int_a^b f'\,dx = f(b) - f(a).$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, there exists $c_i \in [x_{i-1}, x_i]$ such that $f'(c) = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$. Thus,

$$m_f(P, i) \leq \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \leq M_f(P, i),$$

for $i = 1, \ldots, n$. Therefore,

$$\sum_{i=1}^{n} m_f(P, i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_f(P, i)(x_i - x_{i-1}).$$

Now

$$\sum_{i=1}^{n} \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1}) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Thus

$$L_f(P) \leq f(b) - f(a) \leq U_f(P).$$

Since this is true for every partition,

$$\underline{\int_a^b} f'\,dx \leq f(b) - f(a) \leq \overline{\int_a^b} f'\,dx.$$

Since f' is integrable, the upper sum equals the lower sum, so

$$\int_a^b f'\,dx = f(b) - f(a).$$

□

Proposition 24.2. *Let $f : [a, b] \to \mathbb{R}$ be integrable. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f\,dx \geq 0$.*

Proof. Clearly for any partition P, we have $U_f(P) \geq 0$. Thus taking the infimum gives $\overline{\int} f\,dx \geq 0$. But f is integrable, so $\int f\,dx = \overline{\int} f\,dx \geq 0$. □

Proposition 24.3. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable. If $f(x) \leq g(x)$ for every $x \in [a, b]$, then*

$$\int_a^b f\,dx \leq \int_a^b g\,dx.$$

Proof. We see that $(g - f)(x) \geq 0$ for every $x \in [a, b]$, so

$$\int_a^b g\,dx - \int_a^b f\,dx = \int_a^b (g - f)\,dx \geq 0,$$

which implies that

$$\int_a^b f\,dx \leq \int_a^b g\,dx.$$

□

Proposition 24.4. *Let $f(x) = M$ be a constant. Then $\int_a^b f\,dx = M(b - a)$.*

Proof. Let $F(x) = Mx$. Then $f(x) = F'(x)$, and $\int_a^b f\,dx = F(b) - F(a) = Mb - Ma = M(b - a)$. □

Proposition 24.5. *Let $f : [a, b] \to \mathbb{R}$ be integrable and bounded, so that $|f(x)| \leq M$ for some $M > 0$. Then $\int_a^b f\,dx \leq M(b - a)$.*

Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$ be any function. Define a function $f^+ : D \to \mathbb{R}$ by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Also define $f^- : D \to \mathbb{R}$ by

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

- **(a)** $f^- = (-f)^+$;
- **(b)** $f = f^+ - f^-$;
- **(c)** $|f| = f^+ + f^-$.

Proposition 24.6. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $f^+ : [a, b] \to \mathbb{R}$ is also integrable.*

Proof. Let $\epsilon > 0$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ such that $U_f(P) - L_f(P) < \epsilon$. Then for every i we have $M_f(P, i) \geq M_{f^+}(P, i)$, and $m_f(P, i) \leq m_{f^+}(P, i)$; this implies that $M_{f^+}(P, i) - m_{f^+}(P, i) \leq M_f(P, i) - m_f(P, i)$. Thus

$$\begin{aligned} U_{f^+}(P) - L - f^+(P) &= \sum_{i=1}^{n} (M_{f^+}(P, i) - m_{f^+}(P, i))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_f(P, i) - m_f(P, i))(x_i - x_{i-1}) \\ &= U_f(P) - L_f(P) \\ &< \epsilon. \end{aligned}$$

This shows that f^+ is integrable. □

Proposition 24.7. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $|f|$ is integrable, and $|\int_a^b f\, dx| \leq \int_a^b |f|\, dx$.*

Proof. We just saw that f^+ is integrable. Also $-f$ is integrable, so $f^- = (f)^+$ is integrable. Therefore $|f| = f^+ + f^-$ is integrable.

$$\left| \int_a^b f\, dx \right| = \left| \int_a^b f^+\, dx - \int_a^b f^-\, dx \right| \leq \left| \int_a^b f^+\, dx + \int_a^b f^-\, dx \right| = \int_a^b |f|\, dx.$$

□

Observation 4. As a convenience, if $a > b$, then define $\int_a^b f\,dx = -\int_b^a f\,dx$. Then it follows from a previous proposition that

$$\int_a^b f\,dx - \int_a^c f\,dx = \int_c^b f\,dx.$$

Observation 5. If c is constant, then $c = \frac{\int_a^b c\,dx}{b-a}$.

Proposition 24.8. *Let $f : [a,b] \to \mathbb{R}$ be bounded and integrable. Define*

$$F : [a,b] \to \mathbb{R} \quad \text{by} \quad F(x) = \int_a^x f(t)\,dt.$$

Then

- **(a)** *F is uniformly continuous on $[a,b]$;*
- **(b)** *if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof. Since f is bounded, there exits $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$.

Let $\epsilon > 0$ and let $\delta = \epsilon/M$. Let $x, y \in [a,b]$, and suppose that $|x-y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= |\int_a^x f(t)\,dt - \int_a^y f(t)\,dt| \\ &= |\int_x^y f(t)\,dt| \\ &\le M|x-y| \\ &= \epsilon. \end{aligned}$$

Therefore, F is continuous.

Suppose that f is continuous at x_0. Select $\epsilon > 0$; then there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$. Note that this says that for every $t \in [x, x_0]$, we have $|f(t) - f(x_0)| < \frac{\epsilon}{2}$. Compute

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} - f(x_0) \right| \\ &= \left| \frac{\int_{x_0}^x f(t)\,dt}{x - x_0} - \frac{\int_{x_0}^x f(x_0)\,dx}{x - x_0} \right| \\ &= \left| \frac{\int_{x_0}^x (f(t) - f(x_0))\,dt}{x - x_0} \right| \\ &\le \left| \frac{(\epsilon/2)(x - x_0)}{x - x_0} \right| \\ &= \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

□

APPENDIX A

Continuity Examples

Example A.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $x_0 = 2$. Show that f is continuous at x_0.

Proof. Let $\epsilon > 0$; we may assume that $\epsilon < 4$. Let $\delta = \sqrt{x_0^2 + \epsilon} - x_0 = \sqrt{4 + \epsilon} - 2$. Thus $(\delta + 2)^2 = 4 + \epsilon$, so $\epsilon = \delta^2 + 4\delta$.

Suppose that $x \in (2 - \delta, 2 + \delta)$. Then $x + 2 < \delta + 4$, and

$$|f(x) - f(x_0)| = |x^2 - 4| = |x - 2|(x + 2) < \delta(4 + \delta) = \epsilon.$$

□

Example A.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Show that f is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. We wish to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

For simplicity, assume that $x_0 > 0$. Let $\delta = \sqrt[3]{x_0^3 + \epsilon} - x_0$. Solving for ϵ yields $\epsilon = (x_0 + \delta)^3 - x_0^3$.

Let $x \in (x_0 - \delta, x_0 + \delta)$. Then $x > 0$, and

$$\begin{aligned}
|f(x) - f(x_0)| &= |x^3 - x_0^3| \\
&= |x - x_0|(x^2 + x_0 x + x_0^2) \\
&< \delta((x_0 + \delta)^2 + x_0(x_0 + \delta) + x_0^2) \\
&= \delta(x_0^2 + 2x_0\delta + \delta^2 + x_0^2 + x_0\delta + x_0^2) \\
&= \delta(3x_0^2 + 3x_0\delta + \delta^2) \\
&= \epsilon.
\end{aligned}$$

□

Example A.3. Let $f : [0,\infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Show that f is continuous.

Motivation. Graph the curve $f(x) = \sqrt{x}$. Select arbitrary $x_0 \in \text{dom}(f)$. Project up and to the right to find the point $\sqrt{x_0}$ on the y-axis. Draw an ϵ-band around this point. Project the intersection of this band with the graph of f onto the x-axis. Notice that the point on the left of this projection is closer to x_0 than is the point on the right. Let δ be one half of the distance between x_0 and the left endpoint of the inverse image of $[f(x_0) - \epsilon, f(x_0) + \epsilon]$. $\square$

Proof. Let $x_0 \in [0,\infty)$ and let $\epsilon > 0$; wlog assume that $\epsilon^2 \le x_0$. If $x_0 = 0$, let $\delta = \epsilon^2$; clearly this will work. Otherwise set

$$\delta = \frac{1}{2}(x_0 - (\sqrt{x_0} - \epsilon)^2);$$

this is positive. Note that for $x \in \mathbb{R}$, $|x - x_0| = |\sqrt{x} - \sqrt{x_0}|(\sqrt{x} + \sqrt{x_0})$. Then if $|x - x_0| < \delta$, we have

$$\begin{aligned}
|\sqrt{x} - \sqrt{x_0}| &< \frac{\delta}{\sqrt{x} + \sqrt{x_0}} \\
&= \frac{x_0 - (x_0 - 2\sqrt{x_0}\epsilon + \epsilon^2)}{2(\sqrt{x} + \sqrt{x_0})} \\
&= \frac{\epsilon(2\sqrt{x_0} - \epsilon)}{2(\sqrt{x} + \sqrt{x_0})} \\
&< \epsilon\frac{(2\sqrt{x_0} - \epsilon)}{2\sqrt{x_0}} \\
&= \epsilon\left(1 - \frac{\epsilon}{2\sqrt{x_0}}\right) \\
&< \epsilon.
\end{aligned}$$

$\square$

Example A.4. Show that every polynomial function is continuous.

Proof. This is tedious but obviously important. We build it gradually.

Claim 1: The constant function $f(x) = C$, where $C \in \mathbb{R}$, is continuous.
Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = 1$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = 0 < \epsilon$. Thus f is continuous in this case.

Claim 2: The identity function $f(x) = x$ is continuous.
Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = \epsilon$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = |x - x_0| < \delta = \epsilon$, so f is continuous in this case.

Claim 3: The function $f(x) = x^n$ is continuous.
By induction on n. For $n = 1$, the function $g(x) = x$ is the identity function, and so it is continuous. By induction, $h(x) = x^{n-1}$ is continuous. Then by the Continuous Arithmetic Proposition, $f = gh$ is continuous in this case.

Claim 4: The monomial function $f(x) = a_n x^n$ is continuous, where $a_n \in \mathbb{R}$ is constant.
By Claim 1, $g(x) = a_n$ is continuous, and by Claim 3, $h(x) = x^n$ is continuous, so there product $f = gh$ is continuous.

Claim 5: The polynomial function $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous.
By induction on n, the degree of the polynomial.

For $n = 0$, $f(x)$ is constant and therefore continuous.

Assume that $g(x) = a_0 + \cdots + a_{n-1} x^{n-1}$ is continuous. By Claim 4, $h(x) = a_n x^n$ is continuous. Then $f = g + h$ is continuous by the Continuous Arithmetic Proposition. □

Example A.5. Show that every rational function is continuous.

Proof. Let f be a rational function. Then $f(x) = p(x)/q(x)$, where p and q are polynomial functions. Since p and q are continuous, then f is continuous on its domain by a Proposition from the arithmetic of continuous functions. □

Example A.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is discontinuous at every real number.

Proof. Let $x_0 \in \mathbb{R}$. To show that f is discontinuous at x_0, it suffices to find $\epsilon > 0$ such that for every $\delta > 0$, there exists $x \in (x_0 - \delta, x_0 + \delta)$ with $|f(x) - f(x_0)| \geq \epsilon$.

Let $\epsilon = \frac{1}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = 1 > \epsilon$; if x_0 is irrational, let x_2 be a rational in this interval, and we still have $|f(x_2) - f(x_0)| = 1 > \epsilon$. Thus f is not continuous at x_0. □

Example A.7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous at $x = 0$ and discontinuous at all nonzero real numbers.

Proof. Let $x_0 \in \mathbb{R} \setminus \{0\}$; we show that f is discontinuous at x_0. Let $\epsilon = \frac{|x_0|}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = |x_0| > \epsilon$. If x_0 is irrational, let x_2 be a rational in this interval such that $|x_2| > |x_0|$ and we still have $|f(x_2) - f(x_0)| = |x_2| > |x_0| > \epsilon$. Thus f is not continuous at x_0.

Now we consider the behavior of f at zero. Let $\epsilon > 0$ and let $\delta = \epsilon$. Then if $|x - 0| < \delta$, we have $|f(x) - f(0)| = 0$ if x is irrational and $|f(x) - f(0)| = |x|$ if x is rational; in either case, $|f(x) - f(0)| \leq |x| < \delta = \epsilon$, so f is continuous at zero. □

Example A.8. If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Show that f is discontinuous at every rational and continuous at every irrational.

Proof. Suppose that x_0 is rational. We wish to show that f is not continuous at x_0. It suffices to find $\epsilon > 0$ such that for every $\delta > 0$ there exists $x_1 \in (x_0 - \delta, x_0 + \delta)$ with $|x_0 - x_1| > \epsilon$.

Since x_0 is rational, we have $x_0 = \frac{p}{q(x_0)}$ for some $p \in \mathbb{Z}$. Let $\epsilon = \frac{1}{2q(x_0)}$ and let $\delta > 0$. Then $(x_0 - \delta, x_0 + \delta)$ contains an irrational number, say x_1; then $|x_0 - x_1| < \delta$ but $|f(x_0) - f(x_1)| = \frac{1}{q(r)} > \epsilon$. Thus f cannot be continuous at x_0.

Suppose that x_0 is irrational. Let $\epsilon > 0$. It suffices to find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let a be the greatest integer which is less than x_0 and b be the least integer which is greater than x_0; then $b = a + 1$ and $x_0 \in [a, b]$.

For $q \in \mathbb{Q}$, there exist only finitely many points in the set $[a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}\}$ (in fact, this set contains no more than q points). Thus the set

$$D = [a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}, q \leq N\}$$

is finite (there are no more than $\frac{N(N+1)}{2}$ points in this set). Let

$$\delta = \min\{|x_0 - d| \mid d \in D\};$$

since this set is a finite set of positive real numbers, the minimum exists as a positive real number. Then $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Let $x \in (x_0 - \delta, x_0 + \delta)$. If x is irrational, we have $|f(x) - f(x_0)| = 0 < \epsilon$, and if x is rational, we have $|f(x) - f(x_0)| = \frac{1}{q(x)} < \frac{1}{N} < \epsilon$. Thus f is continuous at x_0. $\square$

APPENDIX B

Problem Sets

1. Problem Set A

Problem B.1. Let A, B, and C sets. Show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Problem B.2. Let $f : X \to Y$ be a function.
(a) Show that f is surjective if and only if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.
(b) Show that f is injective if and only if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

Problem B.3. Let X be a set and let $T = \{0, 1\}$. Show that there is a natural bijective correspondence between the sets $\mathcal{P}(X)$ and $\mathcal{F}(X, T)$.

Problem B.4. Let X be a set and let $\mathcal{C} = \{C_1, \ldots, C_m\}$ and $\mathcal{D} = \{D_1, \ldots, D_n\}$ be partitions of X. Define

$$\mathcal{E} = \{C_i \cap D_j \mid C_i \in \mathcal{C}, D_j \in \mathcal{D}\}.$$

(a) Show that $\mathcal{E}$ is a partition of X.
(b) Describe the equivalence relation induced by $\mathcal{E}$ in terms of the equivalence relations induced by $\mathcal{C}$ and $\mathcal{D}$.

Problem B.5. Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$.

Problem B.6. The following statements are true for all $n \in \mathbb{N}$, as can be proved by induction:

- $\sum_{i=1}^{n}(2i - 1) = n^2$;
- $\sum_{i=1}^{n}(4i - 3) = 2n^2 - n$;
- $\sum_{i=1}^{n}(6i - 5) = 3n^2 - 2n$.

(a) State a conjectured generalization of this pattern.
(b) Prove your conjecture.

Problem B.7. Let $b = [3 + \sqrt{2}]^{\frac{2}{3}}$. Show that $b \notin \mathbb{Q}$.

2. Problem Set B

Refer to the following properties of the real numbers:

(F1) $(a+b)+c = a+(b+c)$;
(F2) $a+0=a$;
(F3) $a+(-a)=0$;
(F4) $a+b=b+a$;
(F5) $(ab)c=a(bc)$;
(F6) $a \cdot 1 = a$;
(F7) $a \cdot a^{-1} = 1$ for $a \neq 0$;
(F8) $ab=ba$;
(F9) $(a+b)c = ac+bc$;
(O1) $a \leq a$;
(O2) $a \leq b$ and $b \leq a$ implies $a=b$;
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$;
(O4) $a \leq b$ or $b \leq a$;
(O5) $a \leq b$ implies $a+c \leq b+c$;
(O6) $a \leq b$ implies $ac \leq bc$ for $c \geq 0$.
(C0) every set of real numbers which is bounded above has a supremum.

Problem B.8. Let $a, b \in \mathbb{R}$. Show that $a^2 \leq b^2 \Leftrightarrow |a| \leq |b|$.

Problem B.9. Let $a, b \in \mathbb{R}$. Show that $||a|-|b|| \leq |a-b|$.

Problem B.10. Let S and T be sets of positive real numbers which are bounded above. Suppose that $S \cap T \neq \varnothing$. Show that $\inf S \leq \sup T$.

Problem B.11. Let S be a bounded set of positive real numbers, and let

$$T = \{t \in \mathbb{R} \mid t = s^2 \text{ for some } s \in S\}.$$

Show that T is bounded above, and that $\sup T = (\sup S)^2$.

Problem B.12. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers, and let $A = \{a_n \mid n \in \mathbb{Z}^+\}$. Show that $\lim_{n\to\infty} a_n \leq \sup A$.

3. Problem Set C

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $s, c, q \in \mathbb{R}$.
We say that s is a *limit point* of $\{s_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{Z}^+ \ni n \geq N \Rightarrow |s_n - s| < \epsilon.$$

In this case, we say that $\{s_n\}_{n=1}^{\infty}$ *converges* to s.
We say that c is a *cluster point* of $\{s_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \, \forall N \in \mathbb{Z}^+ \, \exists n \geq N \ni |s_n - c| < \epsilon.$$

In this case, we say that $\{s_n\}_{n=1}^{\infty}$ *clusters* at c.

Problem B.13. Show that $\{s_n\}_{n=1}^{\infty}$ converges, and find the limit.

(a) $s_n = \frac{2^n}{n!}$.
(b) $s_n = \sum_{i=1}^{n} \frac{1}{3^i}$.

Problem B.14. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence which converges to s. Set

$$t_n = \frac{\sum_{i=1}^{n} s_n}{n}.$$

Show that $\{t_n\}_{n=1}^{\infty}$ converges to s.

Problem B.15. Let $s_1 = 1$ and set $s_{n+1} = \sqrt{2s_n}$.

(a) Show that $\{s_n\}_{n=1}^{\infty}$ is bounded.
(b) Show that $\{s_n\}_{n=1}^{\infty}$ is monotone.
(c) Find $\lim_{n\to\infty} s_n$.

Problem B.16. Let $s_n = \sum_{i=1}^{n} \frac{1}{i!}$. Show that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence. (Hint: first show that $\sum_{i=m+1}^{n} \frac{1}{i!} < \frac{1}{m!}$ whenever $m < n$. Proceed by induction on the difference $k = n - m$, which is the number of terms being added.)

Problem B.17. Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $a = \liminf s_n$ and $b = \limsup s_n$. Show that for every $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $s_n \in (a - \epsilon, b + \epsilon)$.

Problem B.18. Define a sequence $\{u_n\}_{n=1}^{\infty}$ by $u_n = 2^k$ if $2^k \leq n < 2^{k+1}$. Define a sequence $\{t_n\}_{n=1}^{\infty}$ by $t_n = n - u_n$. Define a sequence $\{s_n\}_{n=1}^{\infty}$ by $s_n = \sin \frac{2\pi t_n}{u_n}$. Find the set of cluster points of $\{s_n\}_{n=1}^{\infty}$. Justify your answer.

4. Problem Set D

Problem B.19 (Exercise 3.19)**.** Let $f, g : D \to \mathbb{R}$ be uniformly continuous. Show that the function $f + g : D \to \mathbb{R}$ is uniformly continuous. What can be said about the function $fg : D \to \mathbb{R}$? Justify.

Problem B.20 (Exercise 3.20)**.** Let $f : A \to B$ and $g : B \to C$ be uniformly continuous. What can be said about the function $g \circ f : A \to C$? Justify.

Problem B.21 (Exercise 3.23)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* if there exists $h \in \mathbb{R}$ with $h > 0$ such that $f(x + h) = f(x)$ for all $x \in \mathbb{R}$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is periodic and continuous, then it is uniformly continuous.

Problem B.22 (Exercise 3.31)**.** Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous. Let $T = \{x \in [a, b] \mid f(x) = g(x)\}$. Show that T is closed.

Problem B.23 (Exercise 3.44)**.** Suppose that $f : [a, b] \to [a, b]$ is continuous. Show that f has a *fixed point*, that is, there exists $x \in [a, b]$ such that $f(x) = x$.

5. Problem Set E

Problem B.24 (Exercise 4.25)**.** Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Suppose that there exists $M > 0$ such that for every $x \in (a, b)$, we have $|f'(x)| \leq M$.

(a) Show that if $x, y \in (a, b)$, then $|\frac{f(x)-f(y)}{x-y}| \leq M$.
(b) Show that f is uniformly continuous on (a, b).

Hints.
(a) use MVT.
(b) Start like this:
Let $\epsilon > 0$. Set $\delta =$ (an appropriate quantity). We show that if $x, y \in (a, b)$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. □

Problem B.25 (Exercise 4.35)**.** Let $f(x) = x^3 + 2x^2 - x + 1$. Find an equation for the line tangent to the graph of f^{-1} at the point $(3, 1)$.

Observation 6 (Alternate Definition). Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Define

$$Q : \mathbb{R} \setminus \{0\} \to \mathbb{R} \quad \text{by} \quad Q(h) = \frac{f(x_0 + h) - f(x_0)}{h}.$$

Then f is differentiable at x_0 if and only if $\lim_{h\to 0} Q(h)$ exists, in which case $f'(x_0) = \lim_{h\to 0} Q(h)$.

Problem B.26 (Exercise 4.39)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

(1) $f(0) = 1$;
(2) f is differentiable at 0 and $f'(0) = 1$;
(3) $f(x + y) = f(x)f(y)$.

Show that f is differentiable on $\mathbb{R}$ and that $f'(x) = f(x)$ for every $x \in \mathbb{R}$.

Hint. Use the preceding alternate definition of differentiable. □

Definition B.1. A function $f : [-b, b] \to \mathbb{R}$ is called *odd* if $f(x) = -f(x)$ for every $x \in [-b, b]$.

Problem B.27 (Exercise 5.14)**.** Let $f : [-b, b] \to \mathbb{R}$ be an odd function which is integrable on $[-b, b]$. Show that $\int_{-b}^{b} f\, dx = 0$.

Problem B.28 (Exercise 5.27)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Define $h : [a, b] \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Show that h is integrable on $[a, b]$.

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